Deductive Logic

Second Edition

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In memory of
Malcolm Rennie
mentor, colleague, friend

PREFACE

YOU will have gathered from the title page that this is the second edition of a text in deductive logic. While aimed primarily at students undertaking an introductory course in formal logic at either tertiary or upper secondary level, it will also be of use to the general reader who wishes to improve his skills at reasoning and communicating and who already knows or is prepared to discover that working with symbols can be both entertaining and illuminating.

Although the text provides a comprehensive introduction to classical First Order Logic (Propositional Calculus and Quantification Theory), it does not proceed to the theories of identity, definite descriptions and relation-types. We plan to produce a supplementary volume which will cover these extensions to QT as well as other topics in logic e.g., further set theory (finite and transfinite), recursion, modal logic and dynamic logic. It is anticipated that chapters of this supplement will also be available individually. A Teachers' Manual to accompany the text is also planned for subsequent publication.

Major changes and revisions have been made to the first edition (1975). Several new techniques and fresh approaches have been adopted, making the text significantly different from other works in its treatment of the subject matter. Notable features include: employment of possible-world semantics; use of Staines arrows to determine adequacy of formal translation; extensive treatment of logic diagrams; enhancements to standard evaluation methods (e.g., possible-truth tables, method of assigning possible-values; possible-truth trees, the one-tree method, use of decidability theorems for QT); easier development of QT through world-specifications; puzzle-solving heuristics and motivating chapter puzzles; argument modification.

Our method of presentation features top-down approaches, careful distinction between formal and propositional results, an abundance of worked examples, chapter summaries, and thousands of graded exercise questions, with answers. We have devoted considerable space to discussing the relationship between propositions (and arguments) expressed in English and their counterparts in the formal languages. The experience of teaching QT to secondary school students in Queensland indicated that it was preferable to provide a firm foundation in Monadic QT before going on to QT proper: almost all the evaluation techniques can be introduced with monadic predicates, without confronting the complexities of translations involving polyadic predicates.

ACKNOWLEDGEMENTS

Permission from the Board of Secondary School Studies (Queensland) to include problems from past Senior Logic Examination papers (1971-1981) is gratefully acknowledged. The text has made considerable use of the highly original theoretical work by Phillip Staines in the areas of translation adequacy and indirect use of diagrams. Use has also been made of the tree resolution technique of Ian Hinckfuss and the QT translation taxonomy of Corinne Miller. We also wish to record our appreciation for the helpful comments we have received from our students. Finally, we would like to thank our families for putting up with our preoccupation during the writing of this book.

Contents

PART TWO QUANTIFICATION THEORY

1 Introduction

1.1 WHY STUDY LOGIC?

You probably know that logicians delight in the analysis of *arguments*. Such analysis is not only intellectually stimulating but, as we shall see, of great practical importance. Seven arguments are given below: some of these are quite all right and some are not. Try your intuitions out on them now, explaining any defects that you find.

If Raquel is a woman, then Raquel is a person.
Obviously, Raquel is a woman.
So Raquel is a person. (1)

If Fred lives in Queensland then he lives in Australia.
But Fred does not live in Queensland.
Hence Fred does not live in Australia. (2)

Anyone who takes my magic elixir will never need to go to the doctor.
Smith never needs to go to the doctor.
We may infer that Smith takes my magic elixir. (3)

Tom has a blue Holden.
All Holdens are cars.
So Tom has a blue car. (4)

Namu is a small whale.
All whales are mammals.
So Namu is a small mammal. (5)

David is taller than Linda.
Paul is taller than David.
It follows that Paul is taller than Linda. (6)

Nothing is better than chicken casserole.
But dogfood is better than nothing.
It follows that dogfood is better than chicken casserole. (7)

Jot down your own ideas about these before reading on.

Of the first three arguments, only (1) is logically correct. One way of showing an argument is logically defective is to imagine a case where the *conclusion* (the point being argued for) is false even when the other information is correct. For instance, with (2) Fred might live in Tasmania; with (3) Smith might take other medicines or might be naturally very healthy. One reason for studying logic is that it assists us in *drawing appropriate conclusions* from the information available.

Another important benefit deriving from a study of logic is an increased sensitivity to the role that *language* plays in reasoning. The subtle difference in structure between the logically correct argument (4) and the logically incorrect argument (5) may illustrate this point. Whereas the phrase "blue Holden" may be analysed simply as "object which is blue *and* a Holden",the phrase "small whale" means "whale which is small *relative to* whales". Namu might be three metres long, which is small relative to whales but not small relative to mammals.

Many sentences in English are *ambiguous*: they may be read in more than one way. People often misinterpret what another is trying to say. Sometimes two people might be engaged in a heated dispute, not knowing each is trying to say the same thing in different words. People sometimes fall into logical error by sliding from one meaning to another during the course of an argument: occasionally this is done deliberately to trap the unwary: argument (7) is a humorous case in point. Provided we accept as understood the transitive nature of the "is taller than" relation, argument (6) is logically impeccable. Argument (7) is obviously defective because of its preposterous conclusion, but to logically untutored eyes the source of the error may not be obvious because on the surface (7) appears to have the same form as (6). You may have seen that the trouble lies with the first sentence: it could be read as "Chicken casserole is the best food" or as "Having no food is better than having chicken casserole". With the first reading the sentence is plausible; but it is the second, highly implausible reading which is needed to tie the argument into a logically correct structure, since that is the way "nothing" is to be read in the second sentence.

While the ambiguity of English is useful for poetry (conjuring up many images) and jokes (long live puns!), there is no place for it in rational discussion. Within logic a number of special languages have been developed which are *totally unambiguous*. Given a sentence or argument in English, logicians are careful to ascertain its exact meaning; if helpful, they translate it into one of the logical languages before testing it. This is what you will be doing soon. In order to become competent at such translation you will be forced to *stop and think about what the English really means.* Practice at this will help you to both *interpret what others say* and *express yourself clearly*.

With regard to *propositions* expressed in English, logic helps us to not only clarify them but also assess their *structure* and the *relationships* they bear to one another. Consider for instance the three propositions expressed below.

Today is Monday. (8)
Today is Tuesday. (9)
Today is either Monday or not Monday. (10)

Which of these has got to be true under any circumstance? Which two of these could not possibly be true at the same time? From your answers it should be obvious that anyone who believed that (10) was false, or that both (8) and (9) were simultaneously true, would be logically inconsistent. Part of the aim of logic is to help us be as *consistent* as possible in our beliefs. Might one maintain that both (8) and (9) were false and still be consistent?

As will now be apparent, logic is intimately concerned with propositions and arguments. On the one hand it helps us to draw conclusions from a given set of facts (e.g., constructing proofs in mathematics, science, philosophy, everyday life) : on the other hand it assists in spotting errors in reasoning. Thus it facilitates both good reasoning and the detection of bad reasoning. Briefly, logic may be described as the *science and art of* *reasoning correctly.* This does not imply that a person untrained in formal logic is necessarily a poor thinker; one can be a good judo player, for instance, without knowing the physical and physiological principles involved in throwing a person. However there is no doubt that anyone who studies logic with the attitude of applying it in one's everyday thinking will *improve* one's reasoning power and competency at communication (both active and passive). Herein lies the essential relevance of logic. One may treat logic purely as an intellectual discipline and derive much satisfaction from it; indeed, doing symbolic logic is like playing "mathematical games" and we can get a lot of fun out of it. Logic is more than a game however; it is the basis of all fields of rational pursuit. So let's enjoy the game *and* apply it.

1.2 PROPOSITIONS

The term "proposition" is a familiar enough one. As a noun it is often used for "statement", sometimes for "scheme proposed", and sometimes for other things. In logic we give the term a precise meaning and stick to this meaning whenever we use the term. The most important thing about the word "proposition" as it is used in logic was emphasized by the brilliant German philosopher and mathematician Gottfried Wilhelm Leibniz in his short paper "The Nature of Truth" (*c.* 1686):

> I think that this principle is to be sought in the general nature of truths, and that we are to hold to this above all: *every proposition is either true or false.*

If in uttering a sentence a person is actually declaring something to be the case, that something which he asserts is the proposition he is expressing. Some examples should make this clear. Consider the following sentences.

"Earth is a planet." (1)
"John Locke was a philosopher." (2)
"Earth is a star." (3)
"In 1979 Australia was a republic." (4)

Of these, (1) and (2) express true propositions and (3) and (4) express false propositions. What about the next two sentences?

"Earth is the only planet with life on it." (5)
"John Locke loved sailing." (6)

For each of (5) and (6) we do not know whether what is asserted is true, or whether it is false. But we do know it must be either true or false. With (5) for example we know that either

It is true that Earth is the only planet with life on it.

or

It is false that Earth is the only planet with life on it.

Hence (5) and (6) do express propositions. What is expressed is either true or false.

A sentence is declarative or indicative if it declares or indicates that something is the case. If a sentence is not declarative, it doesn't make sense to preface it with "It is true that" or "It is false that"; here are some examples.

"Can you understand this?"
"Let '*L*' stand for 'Logic is great'."
"The green thoughts sipped procrastination."
"Hooray for John Locke!"
"Would that people were more tolerant."
"Please come inside."
"Shut the door."

With each of these cases it is clear that what is expressed is neither true nor false. Thus *some sentences do not express propositions.* The above cases are examples (in order) of the list below.

questions
stipulations
nonsense
exclamations
wishes
requests
commands

While questions, stipulations and nonsense never express propositions, the situation with exclamations, wishes, requests and commands is less straightforward. Consider the following exclamation.

"Logic is fabulous!" (7)

This obviously expresses a true proposition. Now consider the three sentences below.

"Norma wishes you a merry Christmas." (8)
"May you have a merry Christmas." (9)
"I wish you a merry Christmas." (10)

Here (8) reports about a wish but is not itself used to make a wish: it simply expresses a proposition. It doesn't make sense to preface (9) with "It is true that" or "It is false that"; we would usually regard (9) as expressing a wish but not a proposition. It does make sense to preface (10) with "It is true that" or "It is false that": it seems reasonable to say that (10) is used not only to make a wish but also report about it (*cf* (8)): so we could argue that (10) expresses both a wish and a proposition. The analysis of requests and commands is similar to that of wishes. You will notice that a propositional aspect seems to be brought out with these when the speaker refers to himself e.g., "I request that . . .", "I command you to . . .".

What we have said about the logician's use of the term "proposition" may be summarized in the following definition.

Definition: A *proposition* is that which is asserted when a sentence is uttered; it is always true or false (but not both).

Note that *the same proposition may be asserted by different sentences*, e.g.

"Honshū is a Japanese island."
"Honshū is an island of Japan."
"*Honshū wa Nihon no shima desu.*"

As sentences, these are different, but the proposition they express is the same: they "say the same thing in different words".

Note also that *the same sentence may express different propositions* e.g.,

"Today is Monday." (11)
"I am happy." (12)
"Brisbane is in Australia." (13)
"The monk kicked the smoking habit." (14)

The proposition expressed by (11) is different for different days; (12) depends on both the speaker and the time; (13) depends on whether the Brisbane referred to is the capital of Queensland or the town of that name in California etc.; and (14) could mean the monk kicked the habit (garment) which was on fire, or that the monk gave up smoking.

Note on referring to propositions and sentences:

To prevent longwindedness we will frequently, when there is no danger of ambiguity, abbreviate the phrase "the proposition expressed by the sentence (*n*)" to "the proposition (*n*)" or just "(*n*)". With indented examples, we will usually include quotes to indicate we are referring to the sentence inside the quotes, and omit quotes to indicate we are referring to the proposition expressed by the sentence. For instance, sentence (15) expresses proposition (16).

"Logicians like to laugh." (15)
Logicians like to laugh. (16)

You are now ready to start the first exercise. To derive maximum benefit from these questions you should make a serious attempt to provide your own answer before referring to the answers in the back. Any problems of a particularly challenging nature are marked with an asterisk.

NOTES

Some authors use the term "statement" in the way we have used "proposition".

"Rhetorical questions" may express propositions, since they are really assertions disguised as questions for dramatic effect. For instance, a minister of religion who asks rhetorically "What can cause a good man to lose all hope if he believes in a rewarding life after death?" It is really stating that no such person could lose all hope.

Not all logicians would agree with our treatment of the nature of propositions. An overview and useful bibliography on this matter is to be found in "Propositions, Judgments, Sentences, and Statements" by R. M. Gale in *The Encyclopedia of Philosophy* Vol. 6, ed. Paul Edwards (Macmillan, 1967). Two more recent sources worth consulting in this regard are R. Bradley and N. Swartz's *Possible Worlds* (Blackwell, 1979) Ch. 2, and S. Haack's *Philosophy of Logics* (Cambridge U.P., 1978) Ch. 6. For a contrary view see T. J. Richards' *The Language of Reason* (Pergamon, 1978) pp 122–126.

Our inclusion of a propositional aspect for "explicit performatives" such as sentence (10) would not appeal to many logicians. For a clear exposition of how sentences may be used to perform many functions besides stating facts see R. J. Fogelin's *Understanding Arguments* (Harcourt Brace Jovanovich, 1978) Ch. 1 and J. L. Austin's paper "Performative Utterances" which is reprinted as an appendix in Fogelin's text.

Logicians draw a distinction between "tokens" and "types", the former being instances of the latter. In this terminology, the point that "the same sentence may express different propositions" may be rephrased as "tokens of the same sentence-type may express different propositions".

EXERCISE 1.2

1. Which of the following sentences express propositions?

   (a) The baby is laughing.

(b) What's that noise?
(c) Barnard's star may have some planets about it.
(d) 1 + 1 = 3.
(e) Let "I" denote "Inflation is a problem".
(f) Get out of here!
(g) He ran in the race but slipped on a banana peel.
(h) What a glorious day it is!
(i) Santa Clause is jolly.
(j) To fail to achieve the impossible is not to fail.
(k) Please pay attention.
(l) All triangles have four sides.
(m) The next prime minister will be a woman.
(n) Suppose that x is an even number.
(o) What are you thinking?
(p) If it rains then there is moisture in the air.
(q) Super-sausage had a hamburger for lunch.
(r) I wish to be immortal.
(s) Neither circumstances nor criticism will prevent my progress.
(t) Won't you close the door?
(u) Santa Claus is a fictitional character.
(v) Let x mark the spot.
(w) Your wish is my command.
(x) Would that there were peace.
(y) Define "proposition" to mean "bearer of one truth value".

2. For each of the following sentences list at least two propositions that it might be used to express.

(a) John made the mince with his own hands.
(b) The lamb is too hot to eat.
(c) Visiting relatives can be a nuisance.
(d) Students dislike boring lecturers.
(e) Some dogs do not smell.
(f) He is speaking on the subject of old tongues.
(g) The cricket stopped when the bat began to squeak.
(h) Only sons are spoilt.
(i) He addressed the chair from the floor.
(j) Because of the wind the bowler flew off the handle.
(k) Tom the indian would try.

3. Ambiguities often find their way into newspapers. Indicate where they occur in the selection below.

(a) Crash courses are available for those wishing to learn to drive very quickly.
Eastbourne Gazette

(b) A doctor has compiled a list of poisons which children may drink at home.
Ottawa Journal

(c) The man dropped the grammophone while running, but the policeman eventually caught him. It was stated that the defendant had a record.
Belfast Telegraph

(d) At the other end of the building . . . is the section where devotees [of a Hare Krishna group] prepare and eat their meat, fish and egg-free diet of honey-dipped nuts and grains, curried vegetables, yoghurt, milk and rice.
The Sunday Mail

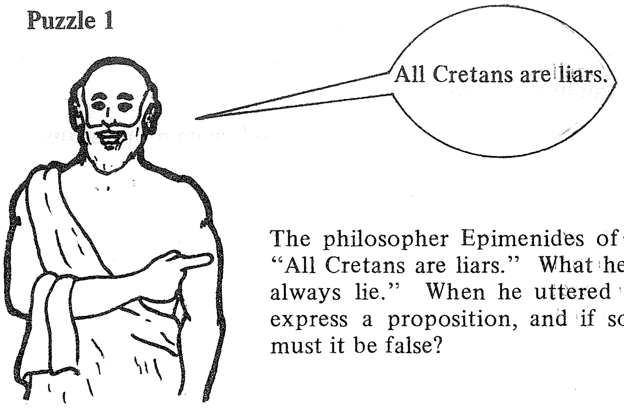
4. Which of the following pairs of sentences may be taken as expressing the same proposition?

   (a) Jack saw Sue. Sue was seen by Jack.
   (b) Vince has a brother. John is Vince's brother.
   (c) Brisbane is south of Mackay. Mackay is north of Brisbane.
   (d) Canning Downs is bigger than Wales. Canning Downs is at least as big as Wales.
   (e) Norma is Selena's mother. One of Selena's parents is Norma.
   (f) Neither John nor Susan is responsible. John is not responsible and Susan is not responsible.
   (g) Seven is larger than five. Five is smaller than seven.
   (h) Adam stood between Brian and Dougal. Between Brian and Dougal stood Adam.
   (i) Yesterday today was tomorrow. Today was tomorrow yesterday.
   *(j) Either the bus has gone or my watch is fast. If the bus has not gone then my watch is fast.

5. Which of the following are true?

   (a) All propositions are true.
   (b) All propositions are true or all propositions are false.
   (c) Every proposition is either true or false.
   (d) If a sentence does not express a truth then it expresses a falsehood.
   (e) No sentence expresses something that is both true and false.
   (f) Every proposition is known to be true or known to be false.
   (g) Some sentences can express different propositions at different times.
   (h) No two sentences in different languages ever express the same proposition.
   (i) "2 + 2 = 4" expresses the same proposition as "twice two are four".
   (j) If a command is obeyed then it is true.

Puzzle 1



The philosopher Epimenides of Crete once said that "All Cretans are liars." What he meant was "Cretans always lie." When he uttered this sentence did he express a proposition, and if so must it be true or must it be false?

1.3 DENIALS

In this and the next few sections we familiarise ourselves with some of the more important types of sentence constructions as well as some key terms which assist in describing and contrasting various types of propositions. We turn first to ways in which a proposition may be *denied*.

The most straightforward way of denying a proposition is to state its *negation*. In English this is usually handled by inserting the word "not". For instance (2) is the negation of (1).

Linda is wide awake. (1)
Linda is not wide awake. (2)

There are many ways in which the negation may be expressed. Each of (3) and (4) is also the negation of (1).

It's false that Linda is wide awake. (3)
It's not the case that Linda is wide awake. (4)

The proposition being negated is called the *negand*. Thus (1) is the negand in (2). In general, if we have some proposition *p*, then *Not p* is the negation of *p*, and in this negation *p* is the negand.

A proposition and its negation form a pair of *contradictory* propositions. This means that in terms of truth or falsehood they must be opposite. For example, (1) and (2) are contradictories. If Linda is wide awake then (1) is true and (2) is false; if she isn't wide awake then (1) is false and (2) is true.

Another way of denying a proposition is to state one of its *contraries*. For instance, each of (5) and (6) is a contrary of (1).

Linda is half asleep. (5)
Linda is sound asleep. (6)

Like contradictories, contraries can't both be true; unlike contradictories however, contraries can both be false. For example, while (1) and (6) can never be true together, if Linda is half asleep they will both be false.

A further example will help clarify things. Consider the following proposition.

Karen is taller than Susan. (7)

(8) is denied by each of the following;

Karen is not taller than Susan. (8)
Karen is shorter than Susan. (9)
Karen is at least 5 cm shorter than Susan. (10)

Of these, (8) is the negation and so a contradictory of (7), and (9) and (10) are each a contrary of (7). Note that (8) and (9) express different propositions. In the situation where Karen and Susan are both the same height (8) will be true and (9) will be false.

Now consider proposition (11).

Karen is shorter than or the same height as Susan. (11)

We do not count this as the same proposition as (8) since (11) contains certain concepts not present in (8). Yet (11) will clearly be true whenever (8) is true and false whenever (8) is false. So (11) is a contradictory, but not the negation, of (7).

Sometimes prefixes are used to make denials. For instance (12) and (13) form a pair of contradictories.

His action was legal. (12)
His action was illegal. (13)

Use of prefixes is not always clearcut, however. Consider the following propositions:

Suzi is popular. (14)
Suzi is unpopular. (15)

Are these contradictories or just contraries? Is it possible for Suzi to be neither popular nor unpopular? What if Suzi is a newcomer who has just joined a class? Does this mean she is currently unpopular with her classmates? Is "unpopular" ambiguous? A sensitivity towards common usage of words is something that all logicians find it necessary to cultivate.

Before getting on to the exercise, let's summarize the main points reached in this section. In this summary, the phrase "can't both be true" means "can't both be true at the same time": this allows that one might be true in one situation and the other might be true in a different situation.

Main Points: *Not p* is the *negation of p*
p is the *negand* in *Not p.*
Contradictory propositions can't both be true, and can't both be false.
Contrary propositions can't both be true, but may both be false.

NOTES

Some authors may wish to include both contradictories and contraries as negations. Our preference has been to treat both as denials, but to classify negation as a special type of contradictory.

We will show later that *any* proposition has just one negation, but has an infinite number of contradictories as well as an infinite number of contraries. In everyday dialogue, the use of the phrase "On the contrary" may be seen as heralding the statement of either a contradictory or a contrary. We choose to define "contrary" in such a way that it must be possible for a pair of contraries to both be false. Thus "contradictory" and "contrary" are mutually exclusive descriptions; in particular, contradictories will not be treated as a subset of contraries.

EXERCISE 1.3

1. Which of the following pairs of propositions are contradictories, and which are contraries?

   (a) Susan is at home. Susan is not at home.
   (b) All men are mortal. No men are mortal.
   (c) All philosophers are fallible. Not all philosophers are fallible.

(d) Brisbane is less than 600 kilometres from Sydney. Brisbane is more than 700 kilometres from Sydney.
(e) No Martians are green. Some Martians are green.
(f) Both Henry and Robert will break the record. Neither Henry nor Robert will break the record.
(g) Either it will rain or there will be a dust storm. It will neither rain nor will there be a dust storm.
(h) Susan fell down. Susan nearly fell down.
(i) Cain and Abel were both young. Cain and Abel were not both young.
(j) Cain and Abel were both young. Abel was not young.
(k) Aristarchus was the first to propose the heliocentric model. Herakleides was the first to propose the heliocentric model.

2. What is the negation of each of the following?

(a) John is sick.
(b) Jack is Bill's brother.
(c) Jack and Jill are both hill-climbers.
(d) Wales is smaller than Queensland.
(e) Jack is Australian and Jill is Scottish.
(f) It never rains.
(g) All men are mortal.
(h) It is possible that you left it in the train.
(i) No fools are rich.
(j) Some students are very wise.

3. State the negation, and a contrary, for each of the following.

(a) That number is positive.
(b) Paul came first in the race.
(c) My favourite recording artist is Donovan.
(d) He is 33 years old.
(e) The universe began with a big explosion 16 billion years ago.
(f) The colour of the car is red.

*4. (a) Prove that for any given proposition *p*, if a contrary of *p* is true then so is each contradictory of *p*.
(b) If a contradictory of *p* is true then what, if anything, may be deduced about the set of contraries of *p*?

*5. Consider the following two propositions:

Logic is very interesting.
Logic is very uninteresting.

(a) Are they a pair of contradictories?
(b) Are they a pair of contraries?
(c) Provide a negation for the first proposition (i.e. negate the true proposition in the pair!).

*6. Explain why the following two propositions are *not* contradictories.

Tom passed the exam.
Tom failed the exam.

7. Which of the following are true?
   - (a) If the first proposition is contrary to the second, then the second is contrary to the first.
   - (b) If the first proposition is contradictory to the second, then the second is contradictory to the first.
   - *(c) If the first proposition is the negation of the second then the second proposition is the negation of the first.

8. (a) Explain why the word "inflammable" was replaced some years ago by the word "flammable". (Hint: Latin prefixes are sometimes ambiguous).
   - (b) In which of the following words is the prefix "in" used for negation: "infamous", "incorrect", "invaluable"? Elaborate.

1.4 CONJUNCTIONS AND DISJUNCTIONS

With the aid of the word "and", any number of assertions can be made in a single English sentence. For instance, both (1) and (2) are asserted by (3).

The bus is gone. (1)
I have no money. (2)
The bus is gone and I have no money. (3)

There are many phrases in English, such as "but" or "although" which we can use instead of "and" to say several things within the one proposition. Each of (1), (2) and (4) are asserted in (5).

My friend will give me a lift. (4)
The bus is gone and I have no money but my friend will give me a lift. (5)

The individual assertions which have been *conjoined* (joined together) in the one proposition are in this context referred to as *conjuncts,* and the overall proposition is termed the *conjunction* of these conjuncts. Thus (3) is the conjunction of (1) and (2); (1), (2) and (4) are the conjuncts in (5).

Sometimes words like "and" are used between nouns, adjectives or other parts of speech. Usually we can rephrase the sentence so that such words lie between sentences. For example, (6) may be reworded as (7). Proposition (6) may thus be viewed as a conjunction of (8) and (9).

John and Vince are acupuncturists. (6)
John is an acupuncturist and Vince is an acupuncturist. (7)
John is an acupuncturist. (8)
Vince is an acupuncturist. (9)

This is not always the case however. Consider the following three propositions.

John and Vince are brothers. (10)
John is a brother. (11)
Vince is a brother. (12)

If (10) means that John is the brother of Vince, then it clearly says more than the conjunction of (11) and (12). Further case studies warning of "logical conjunctivitis" will be discussed in Chapter 2.

With the aid of the word "or", individual propositions may be expressed as alternatives within a single English sentence. For instance, (13) and (14) are listed as alternatives in (15).

Logic is interesting. (13)
Logic is useful. (14)
Logic is interesting or logic is useful. (15)

The alternatives are called *disjuncts* and the overall proposition is said to be a *disjuntion* of these disjuncts. We say the disjuncts have been *disjoined* to form the disjunction. Thus (13) and (14) are disjuncts which have been disjoined to form the disjunction (15).

Disjunctions are usually expressed in English by means of the construction "either ... or ... ", or just "or". Here are some more examples which place the "or" between nouns or adjectives.

Jane is doing either maths or logic. (16)
His favourite colour is red or green. (17)

There are two kinds of disjunction: inclusive and exclusive. *Inclusive disjunction* allows that both disjuncts might be true e.g., when we assert (15) above we should certainly consider it possible that logic is both interesting and useful! Another obvious case of this is (18).

The winner of the logic prize will be either very bright or
very hard working. (18)

When we wish to emphasize that both disjuncts might be true we sometimes add the phrase "or both", as in (19).

Her chubbiness is due to either overeating or lack of exercise or both. (19)

In legal documents this job is performed by the phrase "and/or". A familiar case from mathematics is the following definition:

The union of sets A and B is the set of all elements in either A or B. (20)

Here, elements common to both A and B are included in the union.

Whereas with *inclusive* disjunction we claim merely that *at least* one of the two alternatives is true, with *exclusive* disjunction we claim that *just* one of the two alternatives is true. We will postpone discussion of exclusive disjunctions of more than two alternatives, as complications arise there. Here are some examples of exclusive disjunction.

Jane had either cake or ice-cream but not both. (21)
Terry was born in 1946 or 1948. (22)
Any whole number is either odd or even. (23)

It should be clear that when we exclusively disjoin two alternatives we state that one of the disjuncts is true but definitely not both.

Because it may be used both inclusively and exclusively, "or" may be ambiguous in certain contexts. For the moment, if it is not clear that a disjunction is exclusive we will treat it as inclusive. This simple policy of taking the minimum interpretation is not always appropriate however, as will be discussed in detail in Chapter 7.

Besides being able to detect conjunctions and disjunctions, we should also be able to detect their negations. For example, take the following conjunction:

Jane studies maths and logic. (24)

For this to be true Jane must study both. But there are four possibilities. Jane might study both, or just maths, or just logic, or neither. (24) is true only for the first of the four possibilities. We can set out the four possibilities in a chart.

	Studies logic	*Does not study logic*
Studies maths	(24)	
Does not study maths		

As indicated, (24) is true only in the top left cell.

Now consider (25), which is the negation (and hence a *contradictory*) of (24).

Jane does not study both maths and logic. (25)

(25) is true in all cells other than the top left. It is true in the top right and the bottom left cells because there Jane does not study *both*, only one. It is true in the bottom right cell because there Jane studies neither.

In what cell will (26) be true?

Jane studies neither maths nor logic. (26)

Just the bottom right cell. Propositions (24) and (26) cannot both be true in any cell, but both will be false in the bottom left and top right cells. So (24) and (26) are *contraries*, not contradictories.

Can you see that, given any one cell in the chart, the other three cells *collectively* (i.e. taken together) provide a contradictory to it, but *individually* (i.e. taken one at a time) provide a contrary to it?

Consider now the following inclusive disjunction.

Tom studies either maths or logic. (27)

We can use a chart again to see that (27) is true in three of the four possibilities.

	Studies logic	*Does not study logic*
Studies maths	(27)	(27)
Does not study maths	(27)	

(27) is false only in the bottom right cell, where neither maths nor logic is studied. So a contradictory of (27) is:

Tom studies neither maths nor logic. (28)

Another way of putting (28) is:

Tom does not study maths and he does not study logic. (29)

Exclusive disjunction is more complicated. Take the proposition:

Sue studies maths or logic, but not both. (30)

(30) is true in just two cells, as shown.

	Studies logic	*Does not study logic*
Studies maths		(30)
Does not study maths	(30)	

A contradictory of (30) is:

Sue studies both maths and logic, or she studies neither. (31)

Notice that the contradictories (30) and (31) are true along opposite diagonals of the chart. It should be obvious from the above chart that two contraries to (30) are:

Sue studies both maths and logic. (32)
Sue studies neither maths nor logic. (33)

That's enough about conjunctions and disjunctions for the moment. Let's review the main ideas of this section. To simplify things let us use p and q to denote any two propositions, and assume that "and" and "or" have the senses described earlier.

Main Points: *p and q* is the *conjunction* of *p, q*
p, q are *conjuncts* in *p and q*

p or q is the *disjunction* of *p, q*
p, q are *disjuncts* in *p or q*

p or q or both is the *inclusive disjunction* of *p, q*
p or q but not both is the *exclusive disjunction* of *p, q*

or should be read as *inclusive* unless it is obviously exclusive

Two standard ways of negating a conjunction:–

Not (p and q) : Not both p and q
Either not p or not q

Two ways of negating an inclusive disjunction :–

Not (p or q) : Neither p nor q
Not p and not q

EXERCISE 1.4

1. Identify the conjuncts in each of the following conjunctions.
   - (a) The workmen put down their tools and Brown made a speech.
   - (b) Michael is slow but careful.
   - (c) Alan is here and Betty is here and so is Colin.
   - (d) The gates are not locked and neither the side door nor the back door is closed.
   - (e) The burglar is not in the house but he will be either on the road or on the moors.
   - (f) If anyone is sick they should see the doctor, and it is clear that Bill is not well.
   - (g) If the bus has gone then my watch is slow, and if my watch is slow then the tower clock is slow.

2. In which of the following is "and" used *merely* for conjunction?
   - (a) Jane and Mary are girls.
   - (b) Jane and Mary are sisters.
   - (c) Jane and Mary share a room.
   - (d) Jack is tall and handsome.
   - *(e) Jack and Jill went up the hill.

3. What are the disjuncts in each of the following disjunctions? Also state whether the disjunction is inclusive or exclusive. (Hint: When in doubt treat the disjunction as inclusive)
   - (a) James went either to the library or to the club.
   - (b) Mary is to enroll in either mathematics or physics, but not both.
   - (c) He studied French or logic.
   - (d) The number is either less than 10 or greater than 20.
   - (e) The person who chose that colour scheme was either colour-blind or lacking in aesthetic taste.
   - (f) Either the rain will come and the crop will be planted or we will sell the farm.
   - (g) The number is either not more than 10 or greater than 6.
   - (h) Either Mary takes mathematics and logic or she takes Japanese and computing, but not both.

4. Which of the following pairs of propositions are contradictories and which are contraries?
   - (a) I will go either to Brisbane or to Perth.
     I will go neither to Brisbane nor to Perth.
   - (b) I will go to both Brisbane and Perth.
     I will not go to both Brisbane and Perth.
   - (c) I will go to both Canberra and Cairns.
     I will go to neither Canberra nor Cairns.
   - (d) I will go to both Canberra and Cairns.
     Either I will not go to Canberra or I will not go to Cairns.
   - (e) You will go to Goondiwindi or Gunnedah.
     You will not go to Goondiwindi and you will not go to Gunnedah.

5. In your own words set out the negation of the following.
   - (a) Susan is either a clerk or a teacher.
   - (b) Sandy is both a farmer and an accountant.
   - (c) The bus is slow and time is running out.
   - (d) Either the bus is slow or I am impatient.
   - (e) Both Robin and Chris are mechanics.
   - (f) Cathy is not beautiful but she is attractive.
   - (g) Either you will finish your homework before 9.30 or you will not watch T.V. after 9.30.

1.5 CONDITIONALS AND BICONDITIONALS

The proposition expressed by

"If the clock is slow then we are late." (1)

is a *conditional.* The sentence itself is also called a conditional. Conditionals are so named because they make the following type of claim: on the *condition* that one proposition is true, a second (usually different) proposition is true too. They are often expressed by means of the sentence construction *If. . . then ---*. The sentence immediately preceded by "if" is called the *antecedent*. So, in (1) the antecedent is

"The clock is slow." (2)

The other sentence, preceded by "then", is called the *consequent*. The consequent in (1) is

"We are late." (3)

The propositions expressed by the antecedent and consequent of a conditional sentence are called the antecedent and consequent of the conditional proposition.

There are other ways of expressing a conditional. Instead of sentence (1) we could have

"If the clock is slow we are late." (4)

The "then" is simply left out. It is a bit like the "either" in "either . . . or---". It may often be left off. We can express exactly the same conditional with

"We are late if the clock is slow." (5)

The "if" still precedes the same antecedent, and the other sentence is the consequent. In the same way, the next three sentences express the same conditional proposition.

"If the parcel arrives today then it was posted yesterday." (6)
"If the parcel arrives today it was posted yesterday." (7)
"The parcel was posted yesterday if it arrives today." (8)

In each case the antecedent is

"The parcel arrives today." (9)

and the consequent is

"The parcel was posted yesterday." (10)

In each of the conditionals above, the "if" has marked out the antecedent by preceding it. But there are other ways of expressing conditionals. One way involves the phrase "*only if*".

"The clock is slow only if we are late." (11)

This sentence expresses the same conditional as (1). Similarly,

"The parcel arrives today only if it was posted yesterday." (12)

expresses the same conditional as (6). In these "only if" conditionals the "if" marks out the consequent. So, "Only if" marks out the consequent while "if" by itself marks out the antecedent. We must look to see whether "if" is by itself or with "only". Here are some more pairs of sentences, both expressing the same conditional, one with "if" by itself, the other with "only if".

"If John has ten dollars then John has some money." (13)

"John has ten dollars only if John has some money." (14)
"If John is not at home he is down at the club." (15)
"John is not at home only if he is down at the club." (16)

Every conditional has a *converse*. The converse of

If John has ten dollars then John has some money. (17)

is

If John has some money then John has ten dollars. (18)

We get the converse of a conditional by *swapping the antecedent and the consequent*. The converse of *If p then q* is *If q then p*. The same applies to "only if" conditionals. The converse of

John has ten dollars only if he has some money. (19)

is

John has some money only if he has ten dollars. (20)

It is very important to notice that *a conditional and its converse do not say the same thing*. Can you see the difference?

So, if we want to assert a conditional and its converse, it is no use just asserting the conditional. One way of asserting both is to connect the conditional sentences by the conjunctive "and" e.g.,

"If the set is empty then the set has no members *and*
if the set has no members then the set is empty." (21)

Now consider the following sentence.

"The set is empty *if and only if* the set has no members." (22)

Does (22) express the same proposition as (21)? Well, let's see. It should be clear that (22) expresses the same proposition as (23) does.

"The set is empty *if* the set has no members, *and*
the set is empty *only if* the set has no members." (23)

From the earlier work in this section, we can see that the "if" conditional in (23) expresses the same proposition as

"If the set has no members then it is empty." (24)

and that the "only if" conditional in (23) expresses the same proposition as

"If the set is empty then it has no members." (25)

Hence (22) does express the same proposition as (21). Thus (22) asserts two conditionals. For this reason, any sentence formed from two simpler ones by means of the connective "if and only if" is called a *biconditional*. The proposition expressed by such a sentence is also called a biconditional (it is really a special type of conjunction viz. a conjunction of two conditionals which are converses of each other).

Logicians commonly abbreviate "if and only if" to "*iff*". But when you read "iff" out loud, read it in full as "if and only if".

Main Points: *If p then q* is a *conditional* where *p* is the *antecedent*
and *q* is the *consequent.*
The conditional *If p then q* may also be expressed as:
p only if q

If p, q
q if p
The *converse* of *If p then q* is *If q then p.*
p iff q is a *biconditional.*
p iff q may be expressed as *If p then q, and if q then p.*

EXERCISE 1.5

1. For each of the following conditionals, write down first the antecedent and then the consequent.
   - (a) If taxes are cut people will spend more money.
   - (b) If Snoopy is a dog then Snoopy is an animal.
   - (c) If Tom believes that he is being helped then he is acting in a strange way.
   - (d) Fuzzy is a bear only if she is hairy.
   - (e) Fuzzy is an animal if Fuzzy is a bear.
   - (f) If neither Brown nor Jones breaks the law then they have nothing to fear.
   - (g) The wheat will grow only if it is planted.
   - (h) If it rains then either there will be a flood or the crops will be spoiled.
   - (i) The experiment will not be successful if conditions are not completely sterile.
2. Select those of the following for which both members of the pair express the same conditional.
   - (a) If Sue comes home Bill will be happy.
     Bill will be happy if Sue comes home.
   - (b) If Tiger is a cat then he drinks milk.
     If Tiger drinks milk then he is a cat.
   - (c) If that is a pine then it is an evergreen.
     That is a pine only if it is an evergreen.
   - (d) Albert is consistent if he does not contradict himself.
     Albert does not contradict himself, only if he is consistent.
   - (e) The lights will go on only if there is no power failure.
     There is no power failure only if the lights will go on.
3. Write out the converses of the conditionals in Question 1.
4. Set out the two conditionals conjoined in the following.
   - (a) The number is even if and only if the number is divisible by two.
   - (b) There will be an election if and only if the Governor-General signs the writs.
   - (c) The experiment will be a success if and only if the correct procedures are followed.
   - (d) We will go on a picnic if and only if it doesn't rain.

1.6 BEING CONSISTENT

Consider the following two propositions.

Freddo is a frog. Freddo is green. (1)

It's quite possible for both of these to be true, since Freddo could be a green frog. Any set of propositions which can all be true together is said to be *consistent*. So (1) is a consistent set.

Sometimes we meet a set of propositions which can't all be true at once. Logicians call this an *inconsistent* set. Any pair of contradictories will be inconsistent e.g.,

Freddo is a frog. Freddo is not a frog. (2)

Though we can imagine situations in which either of the two propositions in (2) might be true by itself, it is just not possible that they should both be true together. And the same applies to contraries e.g.,

Tom is taller than Suzy. Tom is shorter than Suzy. (3)

It's impossible to have *both* of these true. Any set of propositions with a pair of contradictories or a pair of contraries will be inconsistent.

Now consider the following case.

John is a philosopher. John has not read Plato's *Republic.* (4)

There is nothing inconsistent about this. It may be unlikely, but nevertheless it is possible that both propositions in (4) are true. So (4) is a consistent set. What about the following set?

If John is a philosopher then he has read Plato's *Republic*.
John has not read Plato's *Republic.* (5)

This set is consistent too. But now let's unite sets (4) and (5):

John is a philosopher.
If John is a philosopher then he has read Plato's *Republic*.
John has not read Plato's *Republic.* (6)

Not all of (6) can be true. Anyone who believed (6) would be logically in error; he would have an inconsistent set of beliefs.

Note carefully that for a set of propositions to be inconsistent it is not generally necessary for *each* of the propositions to be impossible. Consider the individual propositions in (2), (3) and (6) for instance. It is possible nevertheless to have a single proposition which all by itself is inconsistent e.g.,

Today is Monday and not Monday. (7)

Provided we allow the term "collectively" to apply to unit sets, the words "consistent" and "inconsistent" may be accurately interpreted as "collectively possible" and "collectively impossible".

Each one of us should aim for consistency in our web of beliefs. This is not an easy task!

Main Points: A set of propositions is *consistent* iff they can all be true together.
A set of propositions is *inconsistent* iff it's not consistent.

NOTES

The terms "incompatible" and "self-contradictory" are often used instead of "inconsistent", though "incompatible is reserved for sets of at least two propositions, and "self-contradictory" is used mostly with unit sets. Where there are at least two propositions, "consistent" is sometimes replaced by "compatible". One of the first philosophers to make special use of the notion of consistency was Leibniz, who coined the term "compossible" which captures nicely the idea of being possible together.

EXERCISE 1.6

1. Which of the following sets of propositions are inconsistent?
   (a) My car is old. My car is new.
   (b) Some numbers are even. Some numbers are odd. Some numbers are divisible by three.
   (c) If Hitler had invaded England then his army would have taken London. Hitler's

army did not take London. Hitler did not invade England.

(d) If it rains there is high humidity. It is raining. The humidity is not high.

(e) Karen takes either Japanese or Indonesian. Karen does not take Japanese. Karen does not take Indonesian.

(f) Michael takes mathematics and physics. Michael does not take mathematics but he does take physics.

(g) Senator Hall is neither Labor, Liberal nor Independent. Senator Hall is not Country Party. Senator Hall is Independent.

(h) Unless the Parliament stops the Bill it will become law on Tuesday. The Parliament will not stop the Bill. The Bill will become law on Tuesday.

(i) Oranges are fruit. No cats are dogs. All bachelors are unmarried.

(j) The wheat crop will be good only if it rains in July. It rained in July. The wheat crop will not be good.

1.7 ARGUMENTS AND LOGICAL FORM

You may have gathered from § 1.1 that when logicians use the term "argument" they do not mean a heated discussion. A logical argument involves the presentation of evidence or reasons (technically known as *premises*) in support of some point (technically known as *the conclusion*).

Definition: An *argument* consists of a set of propositions, one of which (the *conclusion*) is claimed to follow from the others (the *premises*).

In this book we are exclusively concerned with arguments where the conclusion is claimed to follow with *certainty* (rather than just high probability) from the premises. More will be said about this in the next section.

Before assessing arguments that occur in English ("wild arguments" as Brian Medlin calls them), we need to tame them. This involves separating out the premises and the conclusion, and putting the argument into *standard form*. Consider the following example.

The burglar went out either by the window or by the door.
The burglar did not go out the door, so it follows that
he or she went out by the window. (1)

The phrase "so it follows that" clearly heralds the conclusion, which is:

The burglar went out by the window.

The premises are then the other two propositions:

The burglar went out either by the window or by the door.
The burglar did not go out by the door.

The whole argument may now be written down in standard form as follows:

The burglar went out either by the window or by the door.
The burglar did not go out by the door.

∴ The burglar went out by the window. (1a)

Notice that an unbroken line is used to separate premises from conclusion. The premises are always placed above this line and the conclusion below it. Notice also the use of ∴ as an abbreviation for "therefore", which is always placed in front of the conclusion: this indicates the claim that the conclusion follows from the premises, but is not itself part of the conclusion. Whenever we run across a phrase like "so it follows that", "therefore", "hence", "thus", "so", "clearly" etc. we can be almost sure that the conclusion comes immediately after it. For this reason such phrases are sometimes called "conclusion markers".

There are also "premise markers". Three common ones are "because", "since" and "as". When we come across one of these we can be almost certain that a premise comes immediately after. Sometimes the conclusion comes immediately before one of these premise markers, and sometimes the conclusion comes after the premise which follows this marker. These two situations are illustrated, respectively, in arguments (2) and (3) below.

The figure is a circle, because it's either a circle
or a square, and it's not a square. (2)

The burglar went out either by the window or by the door.
Since the burglar did not go out by the door, he or she
went out by the window. (3)

In standard form, argument (2) becomes:

The figure is either a circle or a square.
The figure is not a square.

∴ The figure is a circle. (2a)

You probably noticed that argument (3) is really the same as argument (1).

Sometimes arguments in English have no obvious conclusion markers or premise markers. But our English intuitions will usually stand us in good stead here. Practice on the exercises in this book will help you tame such arguments. If the proposer of the argument really was unclear in his presentation and if he is available, you should ask him to clarify his argument for you. As a general rule, try to sort out the conclusion first. Then concentrate on the premises.

Once the argument is in standard form we go a step further in our analysis of it. This involves abbreviating sentences which express certain propositions in the argument to single capital letters. The choice of these letters is up to us, but our choice will be easier to remember if we pick the first letter of a key word in the sentence. We set out our choices in a *dictionary*. For example, a suitable dictionary for argument (1a) would be

W = The burglar went out by the window
D = The burglar went out by the door.

Here "=" stands for "is our abbreviation for". Argument (1a) may now be displayed as

$$\begin{array}{l} W \text{ or } D \\ \text{Not } D \\ \hline \therefore W \end{array} \qquad (1b)$$

Notice that we did not abbreviate the first premise to a single letter because it contains simpler propositions (W, D) which occur either independently or in a different surrounding structure elsewhere in the argument. A similar comment holds for the second premise. The upshot of this is that the role played by the key *logical words* (here "or" and "not") is displayed.

Argument (2a) may likewise be exhibited as follows.

Dictionary: C = The figure is a circle
S = The figure is a square

C or *S*
Not *S*
∴ *C* (2b)

You will notice that (1b) and (2b) have a similar pattern or *logical form*. The only difference is in the abbreviated propositions; and the "insides" of these propositions have no bearing on the logical correctness of these arguments. This common logical structure of arguments (1) and (2) may now be shown with the help of small letters like p and q.

p or q
Not q
∴ p

This display is known as an *argument-form*. Because many arguments share common logical forms logicians often conserve energy by focussing their interest on argument-forms rather than treating each individual argument as an entirely new example.

It should be noted that the order in which the premises of an argument are stated is irrelevant. Thus, logical forms of arguments will not be changed merely by changing the order of the premises.

NOTES

In this introductory section we have spoken about just one logical form for each argument. In fact, an argument usually has more than one form and it will be necessary later in the book to take this into account.

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EXERCISE 1.7

1. Pair each of the following abbreviated arguments with another of the same logical form.

(a) *A*
∴ *A* or *B*

(b) If *A* then *B*
Not *B*
∴ Not *A*

(c) Not (*A* and *B*)
A
∴ Not *B*

(d) *C*
∴ *C* or *D*

(e) If *D* then not *E*
If *F* then *D*
∴ *F* only if not *E*

(f) Not *D*
If *C* then *D*
∴ Not *C*

(g) A only if *B*
B only if not *C*
∴ If *A* then not *C*

(h) *D*
Not (*D* and *C*)
∴ Not *C*

2. Each of the following arguments may be paired with one other which has the same logical form. Set each argument out in standard form, using the letters suggested for abbreviation. Then match the pairs. (Not all of these arguments are logically correct).

(a) If Phaedo is a dog then Phaedo is a mammal. If Phaedo is a human then Phaedo is a mammal. Since Phaedo is either a dog or a human, it follows that he is a mammal. (*D*, *M*, *H*)

(b) It will rain only if there is moisture in the air. There is moisture in the air. Hence it will rain. (*R, M*)

(c) If no other site than Lake Pedder can be found for generating power, then Lake Pedder will be flooded. Since no site can be found for power generation other than Lake Pedder, the lake will be flooded. (*N, F)*

(d) John is not enrolled for both Philosophy and Classics. Since he is enrolled for Philosophy, it is clear that he is not enrolled for Classics. (*P, C*)

(e) If Fig. A were of a triangle it would have three sides. But Fig. A does not have three sides. So Fig. A is not of a triangle. (*T, S*)

(f) Either George will apologize and Harold will accept his apology or they will have a prolonged dispute. We will not get both George apologizing and Harold accepting the apology. So they will have a prolonged dispute. (*G, H, D*)

(g) If militants controlled the Union there would be strikes. But there will be no strikes, because militants do not control the union. (*M, S*)

(h) If I am thinking then I exist. Why? Because if I am thinking then it is not possible to doubt that I exist, and if it is not possible to doubt that I exist then I am absolutely certain that I do exist, and if I am absolutely certain that I exist then I do exist. (*T, E, N, C*)

(i) James can know that the theory is adequate only if the theory is, in fact, adequate. So if the theory is, in fact, not adequate then James cannot know that the theory is adequate. (*K, F*)

(j) There will be a good wheat crop only if there is rain. There is rain. Hence there will be a good wheat crop. (*G, R*)

(k) If sample 756 were of copper then it would conduct electricity. But it does not conduct electricity. So it is not copper. (*C, E*)

(l) If Mike were a dog then he would be an animal. But he is not an animal because he is not a dog. (*D, A*)

(m) If salary rises are refused then profits will be cut. The reasons for this are that if salary rises are refused then the union will not call off the strike, and if a strike is not called off by the union then valuable production time will be lost, and if such time is lost then profits will be cut. (*R, C, N, L*)

(n) Brown will not be a member of both the Liberal and Labor Parties. Since he is a member of the Liberal Party it follows that he is not a member of the Labor Party. (*I, A*)

(o) Some actions will count as selfish only if some actions count as unselfish. So if it is false that some actions count as unselfish then it is false that some count as selfish. (*S, U*)

(p) If it rains then the lawn will be watered. If the hose is turned on then the lawn will be watered. Since either it rains or the hose is turned on, it follows that the lawn will be watered. (*R, L, H*)

(q) Either the Prime Minister will resign and the Cabinet will fail to elect a new Prime Minister or the Senate will bring the Government down. We will not get both the Prime Minister resigning and Cabinet failing to elect a new Prime Minister. So, the Senate will bring the Government down. (*R, F, D*)

(r) If no one is willing to volunteer, then we will have to draw lots. Since everyone is unwilling to volunteer, we will have to draw lots. (*V, L*)

1.8 ASSESSING ARGUMENTS

When an argument is proposed in everyday life there are usually two types of claim made (or at least understood). One claim is factual, the other logical. The factual claim is simply that the premises are all true. If even one premise is false, a *factual error* has been committed. Consider the following arguments about the famous Greek philosopher Aristotle.

Aristotle was a man or a woman.
Aristotle was not a woman.
So Aristotle was a man. (1)

Aristotle was Chinese or Greek.
Aristotle was not Greek.
So Aristotle was Chinese. (2)

Argument (1) is free of factual errors, but (2) has a factual error in its second premise. Even one relevant factual error will prevent an argument from establishing its conclusion.

An argument's logical claim is that the premises support the conclusion in a particular way. If this claim is false then a *logical error* has been committed. The logical claim may be for *validity:*

> whenever the premises are true, the conclusion is true

or for *inductive strength:*

> whenever the premises are true, the conclusion is probable

"Probable" here means "likely but not certain". Arguments (1) and (2) make validity claims whereas (3) claims inductive strength.

Almost all galaxies discovered so far, exhibit redshifts.
So probably, the next galaxy discovered will exhibit a redshift. (3)

The word "probably" is not counted as part of the conclusion of (3). Inductive strength is assessed by that branch of logic known as *inductive logic*. Since this text is devoted to *deductive logic*, we will consider only those arguments involving validity claims. *From now on, the term "argument" will be used in this restricted sense.*

Definition: An argument is *valid* iff the truth of the premises guarantees the truth of the conclusion.

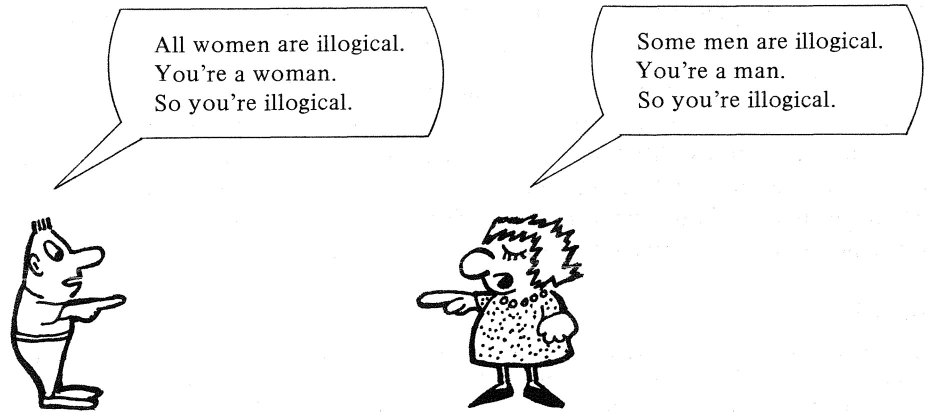
Clearly, argument (1) is valid. Note that validity does not require that the premises are true. An argument is valid iff *just supposing* the premises are true, the conclusion follows. Thus argument (2) is also valid, even though it has a false premise and a false conclusion. Both (1) and (2) have the same logical form: neither commits any logical error.

An argument which is not valid is said to be *invalid*. Here it is possible for the premises to be true without the conclusion being true. Arguments (4) and (5) are both invalid.

Some people are Hindus.
Hence all people are Hindus. (4)

Some dogs are cats.
So all dogs are cats. (5)

Note that argument (1) has no errors, (2) has just a factual error, (4) has just a logical error, while (5) has both factual and logical errors. What kinds of error (if any) are made by Arnold and Bertha in the cartoon below? Has either presented a valid argument?



You should have assessed Arnold's argument as valid (no logical error), but factually in error (1st premise is false). Bertha's argument has no factual errors (all the premises are true), but it is invalid (the conclusion doesn't follow). Because each has made at least one relevant error, each has failed to prove the conclusion argued for.

In order for an argument to establish its conclusion it must have no relevant errors. The only irrelevant errors would be factual errors which have no bearing on the conclusion. We should aim for an argument which is not only valid but which has all its premises true. Such an argument is called *sound*. Sound arguments will always have true conclusions.

Definition: A *sound* argument is a valid argument with all its premises true.

An argument which is not sound is said to be *unsound*. Argument (1) is sound, but arguments (2), (4) and (5) are unsound.

At this point you may be feeling a little uneasy at the way we have been using the term "valid". It doesn't make sense, you might say, that a valid argument can have a false conclusion. If you do feel this then it is probably because you are reading "valid" as "correct", the way it is often used in everyday speech. What you need to realise is that this is *not* the way the word is used in logic. Just as with "proposition" and "argument", the terms "valid" and "sound" are defined in a precise, special way for technical use in logic. The logician's use of "sound" is probably closer to the everyday use(s) of "valid". Note that while premises and conclusion will be true or false, it is incorrect to speak of arguments as being true or false. Arguments are valid or invalid, sound or unsound. Validity is a logical relation between premises and conclusion. With a valid argument, *if* the premises are true then the conclusion will be true too; but if the premises are not all true we have no such guarantee. On the other hand, if the conclusion is true this does not guarantee we have a valid argument: consider the following argument.

Some people are vegetarians.
Therefore Sydney has an opera house. (6)

Here, both premise and conclusion are true, but the argument is invalid because the conclusion does not *follow* from the premise i.e. it is logically possible for some people to be vegetarians without Sydney having an opera house, as it was in 1950.

It is the primary business of logic to examine logical errors (errors in reasoning) rather than factual errors. Nevertheless, since many "facts" are deduced with the aid of reason from other facts, logic helps to reduce factual errors too. And, as we shall see later, logic can detect inconsistencies arising from factual errors.

In formal logic we often assess arguments for validity only. The premises are usually invented simply to provide a specimen exercise, and there is rarely any claim made for their truth. In everyday life however, arguments are used with the intention of establishing their conclusions, and consequently the premises are presented as facts. Thus, everyday arguments should be assessed for soundness: we should question both premises and reasoning i.e. we should search for both factual and logical errors.

NOTES

There is considerable controversy about the difference (if any) between deductive and inductive arguments. Our position is explained more fully in §2.5 of *Inductive and Practical Reasoning* by R. A. Girle, T. A. Halpin, C. L. Miller and G. H. Williams (Rotecoge, 1978).

In this introductory section, the discussion of validity has been somewhat simplified. In particular we have avoided the cases of inconsistent premises and necessary conclusions. A more rigorous treatment which includes these cases will be provided in Chapter 4. An exact definition for validity which makes use of our work on consistency is: An argument is valid iff the set of premises and negated conclusion is inconsistent.

EXERCISE 1.8

1. Describe each of the following arguments by selecting an appropriate letter from the Key provided.

   Key A. No errors
   B. Factual error only
   C. Logical error only
   D. Both logical and factual errors

   (a) Bertrand Russell was a brilliant philosopher
   So Bertrand Russell was a philosopher.

   (b) All cats are animals.
   Therefore all animals are cats.

   (c) Apples are either oranges or lemons.
   But apples are not oranges.
   Hence apples are lemons.

   (d) Some people are vegetarians.
   So some people are not vegetarians.

   (e) Some students are women.
   Some women are koalas.
   So some students are koalas.

2. Which of the arguments in Question 1 are valid?

3. Which of the arguments in Question 1 are sound?

4. Set each of the following arguments out in standard form. Then use your intuitions to decide which are valid.

   (a) If Spinoza was a Queenslander, then he was an Australian.
   Since he was a Queenslander, he was an Australian.

   (b) Hitler was a fascist. Why? Because he just was.

   (c) You can't be both a Christian and a Communist.
   Since you're not a Christian it follows that you're a Communist.

   (d) God is imperfect. Let me tell you why. If the universe is part of God then God is imperfect. But if the universe is not part of God then God is imperfect. And the universe is either part or not part of God.

(e) Queensland is hot, but the Northern Territory is hotter. Obviously the Northern Territory is very hot.

5. Which of the following are true? Where false, give an example.
   - (a) A valid argument must have true premises (i.e. *all* its premises must be true).
   - (b) A valid argument must have a true conclusion.
   - (c) A sound argument must have true premises.
   - (d) A sound argument must have a true conclusion.
   - (e) If a valid argument has true premises it must have a true conclusion.
   - (f) If a valid argument has a false conclusion it must have at least one false premise.
   - (g) If a valid argument has a true conclusion it must have at least one true premise.
   - (h) An invalid argument must have a false conclusion.
   - (i) If an invalid argument has true premises it must have a false conclusion.
   - (j) If the premises are true and the conclusion is false the argument is invalid.
   - (k) If an argument is invalid it must have true premises and a false conclusion.
   - (l) A valid argument may have factual errors but has no logical error.
   - (m) A sound argument has neither factual nor logical errors.
   - (n) An invalid argument must have a factual error.
   - (o) An invalid argument must have a logical error.
   - (p) If the conclusion of an argument is also one of the premises then the argument is invalid.
   - (q) If one of the premises is removed from a valid argument, the resulting argument is invalid.
6. Use your intuitions to assess the validity of the arguments in Exercise 1.7.

1.9 SUMMARY

The art of summarising is very useful. To develop further your own ability to summarise, you should prepare your own chapter summaries before referring to those supplied in this text.

A *proposition* is that which is asserted when a sentence is uttered; it is always either true or false (but not both).
Propositions are usually expressed by sentences in the indicative mood. Some sentences do not express propositions (e.g., all bona fide questions, stipulations, nonsense; some exclamations, some commands, some requests, some wishes).
The same proposition may be expressed by different sentences, and the same sentence may be used to express different propositions.

Not p is the *negation* of *p*, and *p* is the *negand* in *Not p*.
Contradictories can't both be true and can't both be false.
Contraries can't both be true but can both be false.

Given the usual sense of "and" and "or":–
p and q is the *conjunction* of *p, q. p, q* are *conjuncts* in *p and q*.
p or q is the *disjunction* of *p, q. p, q* are *disjuncts* in *p or q*.
p or q or both is the *inclusive disjunction* of *p, q*.
p or q but not both is the *exclusive disjunction* of *p, q*.

or should be read as inclusive unless it is obviously exclusive.

Not (*p* and *q*) may be expressed as: Not both *p* and *q* ; Either not *p* or not *q*

Not (p or q) may be expressed as: Neither p nor q ; Not p and not q

If p then q is a *conditional* where p is the *antecedent* and q is the *consequent.*
If p then q may be expressed as: *if p, q ; q if p ; p only if q*

p iff q is a *biconditional*
p iff q may be expressed as: *if p then q, and if q then p*

A set of propositions is *consistent* iff the propositions can all be true together.
A set of propositions is *inconsistent* iff it is not consistent.

An *argument* consists of a set of propositions, one of which (the *conclusion*) is claimed to follow from the others (the *premises*).
In *standard form* an argument is set out thus:

$$\frac{\text{premises}}{\therefore\ \text{conclusion}}$$

Sentences denoting propositions may be *abbreviated* to capital letters, and *logical forms* of arguments displayed by replacing these letters with p, q, . . . Different arguments may have a common *argument-form.*

Only *deductive* arguments are considered in this book. Here the conclusion is claimed to follow with *certainty* from the premises. If this logical claim is met the argument is *valid*; otherwise it is *invalid*. In everyday life the further claim is made that the premises are all true. If this factual claim is met, and the argument is valid, then the argument is *sound*; otherwise it is *unsound*. Sound arguments will always have true conclusions, but the same cannot be said for valid arguments.

Invalid arguments contain a *logical error.* False premises contain a *factual error.* While formal logic is concerned primarily with logical errors, day-to-day arguments should be searched for both types of error.

Part One

Propositional Logic

2

A New Language

2.1 INTRODUCTION

For most of the next eight chapters our attention focusses on the logical system known as *Propositional Calculus* (PC). The name derives from the fact that we can do calculations in PC to establish *properties of and relationships between propositions, including validity of arguments.* PC is also called "Classical Propositional Logic". Its modern form derives principally from the work of Gottlob Frege (1848-1925) and Bertrand Russell (1872-1970).

This chapter lays the groundwork for Part One by introducing the special logical language used in PC to facilitate its work. We will refer to this as our *Propositional Language* (PL). The other chapters of Part One discuss ways of evaluating propositional formulae, relationships and arguments, and consider both limitations and further applications of PC. Attention is drawn to the connection between propositional reasoning in English and related moves in PC. One chapter is devoted entirely to natural deduction within propositional logic. The final chapter of Part One includes a discussion of the connection between PC and other systems such as set theory and switching calculus. In Part Two, additions are made to PC to form a more powerful system, capable of handling a wider range of propositions and arguments.

Some students may find the first half of this chapter a little difficult because of the abstract way in which it is developed. We ask these people to make a patient effort, as there is a good reason for structuring the content in this way; the point of the symbolic game will soon be made clear. The reader will gain an insight into the structure of this chapter if he follows the discussion in the next paragraph.

Any language contains a set of *symbols* (e.g., English contains the letters "a", "b", etc.) and you will say hello to some interesting new symbols (e.g., "⊃", " ≢ ") in the course of learning PL. The general study of symbols is called *semiotic* and this may be roughly divided into three sections as shown.

SEMIOTIC
- SYNTAX (study of symbols as uninterpreted *objects*)
- SEMANTICS (study of the *meanings* of symbols)
- PRAGMATICS (study of the *intended use* of symbols)

We begin by studying *syntax*, playing around with strings of symbols but not reading anything into them (i.e., treating them as nothing more than marks on paper). Then we move on to *semantics* where we now give the marks some *meaning*: in particular we will talk about the symbols *denoting* (or "standing for") something and, most importantly, we will be concerned with *truth* and *falsity* (since certain strings of symbols will denote propositions). Having done all this we will then find out how to use the symbols to our advantage: this is *pragmatics*.

NOTES

Propositional Calculus is also known as Sentential Logic, Truth-functional Logic, or 2-valued Logic.

An ancient form of propositional logic was developed in the third century B.C. by the Stoic philosophers, especially Chrysippus (280-205 B.C). George Boole (1815-1864) in his *Laws of Thought* (1847) developed an algebra which is structurally in agreement with PC. Frege's seminal paper, the *Begriffsschrift*, was published just over a century ago (1879). Russell's major works on logic were *The Principles of Mathematics* (1903) and, with his former teacher Alfred North Whitehead as co-author, *Principia Mathematica* (1910-1913).

2.2 SYNTAX

To get under way we write down a list of the *primitive symbols* that make up PL and name any that are unfamiliar. They are called "primitive" since they are not defined in terms of anything else.

Primitive Symbols:

p, q, r, s, t	small letters, from p to t, with or without subscripts
(,)	left and right parentheses
$\sim$	tilde
&	ampersand
$\vee$	wedge
$\supset$	hook
$\equiv$	tribar
$\not\equiv$	slashed tribar

In English some combinations of words are counted as grammatically correct sentences, e.g.,

The cat sat on the mat

whereas others are not, e.g.

mat cat sat the on the.

Likewise in PL certain strings of symbols will constitute *well formed formulae*, e.g.,

$(p \,\&\, q)$

and others will not, e.g.,

$pq($

To save a bit of writing we will introduce the abbreviation "wff" (pronounced "woof") for 'well formed formula". Also we will use the Greek letters α (alpha) and β (beta) to represent wffs in general. A well formed or grammatically correct sentence in English is one that obeys the rules of English grammar; analogously, a well formed formula in PL is one that obeys the *formation rules* of PL.

Formation Rules:

Basis Clause:	p, q, r, s, t taken individually, are wffs	(B)
Recursive Clauses:	If α is a wff, so is $\sim\alpha$.	(R $\sim$)
	If α and β are wffs, so is $(\alpha \,\&\, \beta)$	(R &)
	" " $(\alpha \vee \beta)$	(R $\vee$)
	" " $(\alpha \supset \beta)$	(R $\supset$)
	" " $(\alpha \equiv \beta)$	(R $\equiv$)
	" " $(\alpha \not\equiv \beta)$	(R $\not\equiv$)
Terminal Clause:	If α is a wff, it is so because of the above rules.	(T)

In building up Wffs, the basis clause gives us something to start with, the recursive clauses allow us to make longer and longer wffs, and the terminal clause prevents us from writing down just anything and calling it a wff. The names of the rules are shown in the right hand column. Constructing wffs from the rules is fun. We use *assembly lines*. These are just like assembly lines in factories, but we construct formulae of PL out of the primitive symbols by using the formation rules.

Example:

1.	p	B
2.	$\sim p$	1, R $\sim$
3.	$\sim\sim p$	2, R $\sim$
4.	q	B
5.	$(\sim\sim p \vee q)$	3, 4, R $\vee$
6.	$((\sim\sim p \vee q) \equiv q)$	5, 4, R $\equiv$
7.	$\sim((\sim\sim p \vee q) \equiv q)$	6, R $\sim$

You will notice a column of working on the right. This shows the *justification* for each step by *quoting the lines and rules used*. Before going any further, have a go yourself at generating some well formed formulae, and include a justification column beside your assembly line.

The rules enable us to decide whether or not a formula is a wff. If it can be constructed from the rules it is a wff; if it can't, it is not a wff. The best way to understand this is to work through some problems, checking your answers and referring back to the rules if you make a mistake.

The following six symbols of *PL* are known as *operators:* $\sim$, &, $\vee$, $\supset$, $\equiv$, $\not\equiv$. The reason for calling them this will be explained in the next section. You should be able to see from the formation rules of *PL* that every wff with more than one symbol in it will have at least one operator. Now the *last operator added* in building up a wff is called the *main operator* of the wff. We indicate the main operator by placing an arrow underneath e.g.,

$$\underset{\uparrow}{\sim} p \qquad (p \underset{\uparrow}{\vee} q) \qquad \underset{\uparrow}{\sim}(p \supset q) \qquad \underset{\uparrow}{\sim}\sim p$$

Quite often, different assembly lines may be used to construct the same wff. For instance, with the assembly line example above, step 4 could have been done before step 2. For any given wff however, all assembly lines will have the same operator added for the last step. Thus the formation rules of *PL* ensure that each wff with more than one symbol has a *unique* main operator. Hence, regardless of how we build up $\sim((\sim\sim p \vee q) \equiv q)$ its main operator will be its left-most $\sim$.

NOTES

If we ever need more small letters in *PL* than just *p, q, r, s, t* we may use subscripts with these e.g., $p_1, p_2, \ldots, q_1$, etc.

‘ Wff” may also be pronounced “wif” but we have been informed by Snoopy that “woof” is preferable. For the rest of Part One, “wff” will be taken to mean “wff of *PL*”.

EXERCISE 2.2

1. Which of the following are wffs (of PL)?

   (a) $p \sim$
   (b) (p)
   (c) $(p \mathbin{\&} q)$
   (d) $p \mathbin{\&} q$
   (e) $p \mathbin{\&} q)$
   (f) $\sim\sim\sim(p \mathbin{\&} p)$
   (g) $(p \sim p)$
   (h) (pq)
   (i) $p \vee (q \vee r)$
   (j) $(p \vee (q \vee r))$
   (k) $(p \vee (\sim q \vee r))$
   (l) $\sim(p \not\equiv q)$
   (m) $(p \supset \supset q)$
   (n) $\supset p$
   (o) $(p \equiv (q \equiv (r \equiv s)))$
   (p) $((((p \mathbin{\&} (q \vee (r \supset (s \equiv t))))$

2. For each of the following assembly lines fill in the correct justification for each step, and indicate the main operator in the final wff.

   (a) 1. p
   2. $\sim p$
   3. $(p \mathbin{\&} \sim p)$
   4. $\sim(p \mathbin{\&} \sim p)$

   (b) 1. p
   2. q
   3. $\sim p$
   4. $\sim q$
   5. $(\sim p \mathbin{\&} \sim q)$

   (c) 1. p
   2. q
   3. r
   4. $(p \vee q)$
   5. $(q \supset r)$
   6. $((p \vee q) \equiv (q \supset r))$
   7. $\sim((p \vee q) \equiv (q \supset r))$

   (d) 1. p
   2. q
   3. $(p \not\equiv q)$
   4. $\sim p$
   5. $\sim q$
   6. $(\sim p \mathbin{\&} \sim q)$
   7. $(p \mathbin{\&} q)$
   8. $((\sim p \mathbin{\&} \sim q) \vee (p \mathbin{\&} q))$
   9. $((p \not\equiv q) \not\equiv ((\sim p \mathbin{\&} \sim q) \vee (p \mathbin{\&} q)))$

3. Generate the following wffs from the formation rules, showing the justification for each step.

   (a) $\sim\sim\sim p$

(b) $(p \& (q \vee r))$
(c) $(\sim p \supset (q \equiv \sim q))$
(d) $((p \not\equiv q) \& (q \vee r))$
(e) $(\sim(p \equiv \sim q) \supset \sim\sim(\sim p \equiv q))$

*4 Although our language *PL* has been constructed with a definite interpretation in mind, the syntax of a language may be discussed whether or not the symbols are later to be given any meaning. In this question and the next we have invented a couple of languages with no interpretation in mind.

A new language *DL* is defined as follows:

Primitive Symbols: △ □ ★ ⊥

Formation Rules:
⊥ is a wff (B)
If α is a wff, so is α□□ (R□)
If α is a wff, so is α△ (R△)
If α and β are wffs, so is ★α★β (R★)
If α is a wff, it is so because of the above rules. (T)

(a) State whether or note the following are wffs of *DL*. (Answer Yes or No)

(i) ⊥□□△
(ii) ★△⊥★⊥□□
(iii) ★⊥△△△★⊥△□□

(b) Generate the following wff from the formation rules of *DL* quoting the line and rule used for each step.

★★⊥★⊥△△★⊥□□

*5 A new language *TL* is defined as follows:

Primitive Symbols: □ ↑ ○ ★

Formation Rules:
□ is a wff (B)
If α is a wff so is ○α○ (R○)
If α and β are wffs so is ★$\alpha\beta$ (R★)
If α and β are wffs so is α↑↑β (R↑)
If α is a wff, it is so because
of the above rules. (T)

(a) Which of the following are wffs of *TL*? (Answer Yes or No)

(i) ○↑↑○
(ii) ★□○□○
(iii) ○★□□○↑↑□

(b) Generate the following wff from the formation rules of *TL*, quoting the lines and rules used for each step.

★○□○↑↑★○□○□□

2.3 SEMANTICS

So far we have not interpreted the symbols of PL. We now give meaning to these symbols by providing definitions. To facilitate understanding of the new concepts, a comparison will be made with familiar ideas from mathematics.

In algebra the letter x is often used as a (numeric) *variable*. Consider for example the expression

$$x(x + 1) = x^2 + x.$$

This equation is true no matter what number we substitute for x. For instance, putting 3 for x we have

$$3(3 + 1) = 3^2 + 3$$

and putting 5 for x gives

$$5(5 + 1) = 5^2 + 5.$$

Note that while we are free to choose any value for x, the same value must be substituted for every occurrence of x in the expression. For example, the following equation (obviously incorrect) would *not* count as an instance of the algebraic expression above.

$$3(5 + 1) = 9^2 + 4.$$

Similarly, logic uses the small letter p (or q, r, s, t) as a (*propositional*) *variable* to denote any proposition. For example the expression

If p then p

is true no matter what proposition we substitute for p, e.g.,

If I am a man then I am a man.
If logic is marvellous then logic is marvellous.

In keeping with the notion of a variable, when a substitution is made, each p in the expression must be replaced by the *same* proposition; thus

If I am a man then logic is marvellous

does *not* count as an instance of "If p then p". It should be noted that propositional variables range over complex propositions too, e.g.,

If I am happy and you are happy then I am happy and you are happy.

is an instance of the expression above.

Definition: A *propositional variable* (PV) stands for any proposition, and is represented by a small letter in the range p, q, r, s, t. In substitution, every occurrence of the PV in the expression should be replaced by the *same* proposition.

Having dealt with the first item on our list of primitive symbols we now move on to the next: *parentheses*. In this case it will be to our advantage to incorporate the pragmatic aspect. In logic, parentheses (,) have the same meaning as in mathematics: *an expression in parentheses is to be evaluated before operating on it from the outside*. This convention allows us to discriminate between algebraic formulae like

$x + (y \text{ x } z)$
$(x + y) \text{ x } z$

and logical expressions like

p and (q or r) (1)
(p and q) or r (2)

Here (1) asserts that p is true and that at least one of q or r is true: (2) asserts that either p and q are both true or r is true. Clearly, the expression

p and q or r (3)

is ambiguous: it might be read as either (1) or (2). Let us consider an example in English.

"Earth is a star and Venus is a star or Sirius is a star." (4)

As it stands, this sentence is ambiguous. It could mean

(Earth is a star and Venus is a star) or Sirius is a star. (5)

which is true, since Sirius is a star; or it could mean

Earth is a star and (Venus is a star or Sirius is a star). (6)

which is false, since Earth is not a star. In written English, a *comma* is often used in place of a parenthesis. For instance (5) would be expressed as

"Earth is a star and Venus is a star, or Sirius is a star." (7)

In spoken English a *pause* does the job of a comma. English sentences may also be disambiguated by rephrasing, but sometimes it is extremely difficult to prevent ambiguities from creeping in. One of the nice things about *PL* is that its rules for adding parentheses ensure that any wff may be read in only one way i.e. *the formation rules of PL prevent such ambiguities from occurring.*

It should be realised however that there are often cases where parentheses are redundant, e.g.,

(q or r) (8)
p and (q and r) (9)

In (8) and (9) the meaning would be unaltered by the deletion of the brackets. Although our formation rules insisted on extra parentheses whenever another propositional variable was added to a formula, we shall, for the sake of simplicity in reading and writing formulae, allow this rule to be modified by the following agreement.

Practical Concession: Parentheses may be dropped where no ambiguity results.

One immediate consequence of this is that *outer-most parentheses may be omitted from any formula.* Note that while parentheses may be omitted around "q or r" in (8), they must be inserted before incorporating this into an expression like (1); otherwise we will end up with something like (3) again.

In practice we will also allow, for the sake of clarity, any form of brackets to be used, e.g., [ ], rather than just parentheses (i.e., round brackets). Thus the formula

$$\sim((p \supset q) \supset (r \supset (q \supset r)))$$

may be replaced by the easier to read equivalent

$$\sim[(p \supset q) \supset (r \supset (q \supset r))] \tag{10}$$

Where helpful, different *colours* may be used for different pairs of matching brackets. For instance, the structure of the above formula would be more obvious if we used a different colour for the parentheses in $(q \supset r)$.

In addition, we will occasionally make use of a *dot notation.* In this book our main use of dots will be to highlight the main operator in certain important formulae. In the example below, the original formula is made more readable by first deleting the outermost parentheses and then introducing dots.

$$(((p \mathbin{\&} q) \supset r) \equiv (p \supset (q \supset r)))$$
$$((p \mathbin{\&} q) \supset r) \equiv (p \supset (q \supset r))$$
$$(p \mathbin{\&} q) \supset r \;.{\equiv}.\; p \supset (q \supset r)$$

The expressions to which an operator is added in an assembly line are known as the *operands* of that operator. In the above example the operands of $\equiv$ are $((p \mathbin{\&} q) \supset r)$ and $(p \supset (q \supset r))$. Notice above that as dots were placed around an operator, the outer parentheses of each of its operands were removed.

In some cases it will be handy to use dots to highlight the main operator of *sub-formulae.* For example, formula (10) may be replaced by

$$\sim [p \supset q \,.\!\supset\!.\, r \supset (q \supset r)]$$

By use of multiple dots, the dot notation may be extended to completely eliminate the need for brackets. There are several dot notations extant, and the most popular of these are discussed in Chapter 9.

Practical Concession: Any form of brackets (including dots) may be used instead of parentheses.

Let us now consider the six remaining symbols in our list of primitives (viz. $\sim$, &, $\vee$, $\supset$, $\equiv$, $\not\equiv$). As you know, these are termed operators. More exactly, they are called *propositional operators.* You are already familiar with several algebraic operators, e.g., +, – (unary); +, –, x, ÷ (binary). The unary minus "–" operates on a single number (e.g., 5) to form another number (–5), the binary multiply "x" operates on two numbers (e.g., 2, 3) to form another number (2 x 3, i.e. 6). We might refer to these algebraic operators as "number forming operators on numbers"; in like fashion, our logical operators may be described as "proposition forming operators on propositions". $\sim$ is different from the other propositional operators in being *monadic*: it operates on a *single* proposition (e.g., p) to form another proposition ($\sim p$), cf. unary +, –. The others (e.g., &) are *dyadic*, operating on *two* propositions (e.g., p, q) to form a single proposition (p & q), cf. binary +, –, x, ÷.

Before learning any more about our operators it will be necessary to make a brief detour through some related concepts. In chapter 1 we saw that a proposition must be true or false (but not both). Another way of saying this is that a proposition must have a *truth value* of *1* or *0* (but not both).

Definition: There are two *truth values*: TRUE (denoted by 1)
FALSE (denoted by 0).

We are now in a position to give *meaning* to the six operators in *PL. The operators are defined by their truth tables.* What are truth tables? Well, the best way to answer this question is to show you some. Here is the truth table for $\sim$.

p	$\sim p$
1	0
0	1

$\sim p$ has the opposite truth value to p

As we know, 1 and 0 stand for "true" and "false" respectively. You will notice that this table has two rows of truth values (rows are always horizontal) and two columns of truth values (columns are always vertical). The first row of values says that given any proposition p which is true, then $\sim p$ will be false. The second row says that when p is false, $\sim p$ is true. In other words, $\sim p$ has the opposite truth value to p. Strictly, it is incorrect to speak of PVs as being true or false. However we will often speak of truth values being assigned to PVs to indicate generally the result of substituting propositions for those PVs.

The section of the table below the heading line and to the left of the double vertical line is called the *matrix* of the truth table: it *lists all the permutations of truth values for the propositional variables in the formula.* When only one PV is involved there are only the two cases:

p
1
0

When two PV's are involved however, there are four permutations:

p	q
1	1
1	0
0	1
0	0

This matrix may be used to define all the dyadic operators.

p	q	$p \& q$
1	1	1
1	0	0
0	1	0
0	0	0

$p \& q$ is true iff both p and q are true

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

$p \vee q$ is true iff at least one of p, q are true

p	q	$p \supset q$
1	1	1
1	0	0
0	1	1
0	0	1

$p \supset q$ is false iff p is true and q is false

p	q	$p \equiv q$
1	1	1
1	0	0
0	1	0
0	0	1

$p \equiv q$ is true iff p and q have the same truth values

p	q	$p \not\equiv q$
1	1	0
1	0	1
0	1	1
0	0	0

$p \not\equiv q$ is true iff p and q have opposite truth values

Though we have used the propositional variables p, q in defining the operators, this has been for convenience rather than necessity. Often we represent propositions by wffs which are more complicated than simple PVs, in order to show the relevant structure of the propositions (see §2.4). So $\supset$, for instance, may be defined as follows, where α and β are any two wffs:

$\alpha \supset \beta$ is false iff α is true and β is false

An alternative way to picture the definition of $\supset$ is:

$1 \supset 1 = 1$
$1 \supset 0 = 0$
$0 \supset 1 = 1$
$0 \supset 0 = 1$

For simplicity, "$1 \supset 1 = 1$" may be read "true hooks true, is true"; but this reading should be understood as merely an abbreviation for "(any proposition consisting of) a true proposition hooking a true proposition, is itself a true proposition". A similar comment applies to the other three lines of the definition.

An even shorter way of writing the definition for $\supset$ is to use a *Cayley table* as shown below.

$\supset$	1	0
1	1	0
0	1	1

i.e. $1 \supset 1 = 1$ $1 \supset 0 = 0$
$0 \supset 1 = 1$ $0 \supset 0 = 1$

In Cayley tables, the left operand is represented underneath the operator and the right operand is represented on the right of the operator. The values in the body of the table show the results of the operation being carried out between the operands on that particular row and column.

Similarly, the other operators may be defined without using the symbols *p* and *q*.

While you will need to learn the definitions of the operators in *PL*, it will be easier to remember them if you can associate the operators with the English expressions they are used to translate. The next section on pragmatics will cover this. So do not bother to memorize the definitions by rote at this point.

NOTES

We have included one monadic and five dyadic operators in *PL*. These are more than adequate for most applications of propositional logic. Infinitely more propositional operators however could be defined. There are 4 monadic propositional operators, 16 dyadic operators, 256 triadic operators, and in general $2^{(2^n)}$ *n*-adic operators. For a survey of propositional operators see Ch. 9.

While from the point of view of syntax, the symbols p, q, r, s, t, (,), $\sim$, &, $\vee$, $\supset$, $\equiv$, $\not\equiv$ are primitive, from the point of view of semantics they are no longer primitive since they have been defined in terms of other things. It is possible to divide semantics up into *formal semantics* (where for instance the operators would be defined in terms of the values 1 and 0 but no interpretation would be given to 1 and 0) and *informal semantics* (where for instance 1 and 0 are interpreted as the values *true* and *false*). The dividing lines between syntax, semantics and pragmatics are sometimes drawn differently by different logicians.

In metatheory a distinction is drawn between the *use* and the *mention* of symbols. For instance, the word "Australia" is used in the first sentence below but mentioned in the second.

"Australia has fourteen million people."
" 'Australia' has nine letters."

While quotes are often used in written English to disambiguate between use and mention, we will frequently expect the reader to determine, from the context, in which of these two ways the symbols of *PL* are being employed. For instance, in defining the propositional operators, the symbols *p* and *q* were being *used* to stand for any proposition rather than just being mentioned as symbols.

The semantics we have adopted are classical 2-valued semantics. It should be noted that non-classical semantics exist which include more than two values (e.g., 0, 1, 2) or combine values (e.g., $\{0\}$, $\{1\}$, $\{0, 1\}$).

EXERCISE 2.3

1. Render each of the following formulae more readable by deleting outermost parentheses and making use of alternative brackets or dots.

(a) $((p \supset q) \equiv (\sim p \vee q))$
(b) $((p \,\&\, (q \vee p)) \supset ((q \,\&\, q) \supset p))$
(c) $((p \supset q) \supset (((q \supset r) \,\&\, (r \supset s)) \supset (p \supset s)))$

2. Eliminate any dots and alternative brackets from the following formulae in favour of parentheses, and insert outermost parentheses.

(a) $p \,\&\, q\ .\!\equiv\!.\ q \,\&\, p$
(b) $(p \supset q) \,\&\, p\ .\!\supset\!.\ q$
(c) $\sim [p \,\&\, (q \vee r)\ .\not\equiv.\ (p \,\&\, q) \vee (p \,\&\, r)]$

3. Use the definitions of the propositional operators to complete the truth values of $\sim p$, $\sim q$, $q \supset p$, and $p \supset p$ in the table below.

p	q	$\sim p$	$\sim q$	$q \supset p$	$p \supset p$
1	1				
1	0	0	1	1	1
0	1				
0	0				

To calculate the values for each row, look across to the values of p and q in that row of the matrix, and then use the operator definitions. As a hint, the second row is filled in already; the values of the four formulae were obtained as follows. On row 2, p is 1 and q is 0. So the value of $\sim p$ is ~ 1 which = 0. The value of $\sim q$ becomes ~ 0 which = 1. Next, $q \supset p = 0 \supset 1$ which = 1. Finally, $p \supset p = 1 \supset 1$ which = 1.

4. Construct Cayley tables for $\&$, $\vee$, $\equiv$ and $\not\equiv$.

5. The dyadic operator $\subset$ is defined as follows: $p \subset q$ is false iff p is false and q is true. Set out a truth tabular definition for $\subset$.

2.4 TRANSLATION BETWEEN LANGUAGES

Having developed a new logical language it is about time we started to *use* it to help us analyse ordinary arguments, which after all occur in English. The first thing to do is to find out how English sentences may be translated into (or "mapped onto") formulae of PL which express the same propositions. This is precisely the task to which we address ourselves in this section.

In algebra, many general results about numbers may be stated in terms of numeric variables e.g., $x + y = y + x$. In arithmetic however, when dealing with particular numbers we find it convenient to introduce numerals which constantly designate the same value e.g., $1 + 2 = 2 + 1$. Likewise in formal logic, many general results about propositions may be stated in terms of propositional variables e.g., $p \supset p$ is seen to be true for all instances of p.

p	$p \supset p$
1	1
0	1

But when translating propositions and arguments given in English we are dealing with *particular* propositions, and it is convenient to denote these by propositional *constants*. These were introduced informally in §1.7, but it is now time to lay down an exact definition.

Definition: A *propositional constant* is a capital letter used in translation to stand for a particular proposition; as an aid to memory we usually pick the first letter

of an important word in the sentence, e.g.,

S = Selena is beautiful

When used for translation purposes during a particular example, a propositional constant will designate the same proposition throughout. In another example however, the same constant may be used to designate another particular proposition. For instance, in one context we might stipulate

A = Apples are delicious

and in quite a different context we might stipulate

A = The student is finally awake.

Propositional constants may thus be thought of as "contextual constants" (i.e. they are constant within a given context).

Although we could handle the PC analysis of particular propositions and arguments by regarding them as *instances* of various *forms*, it certainly makes life easier if we include propositional constants as part of our propositional language. This we now do.

Practical Concession: Propositional constants A, B, C, etc will be allowed to feature in the formulae of PL. A propositional constant standing alone will be treated as a wff.

At this stage it is also convenient to draw a distinction between simple (or atomic) propositions and compound (or molecular) propositions. Roughly speaking, *a proposition is* ***atomic*** *if it contains no other proposition; otherwise it is* ***compound***. Here are some atomic propositions:

Logic is easy. (1)
The cat sat on the mat. (2)

Here are some compound propositions:

If I persevere I will understand this. (3)
Today is not Monday. (4)
Today is Monday or today is Tuesday. (5)
Today is either Monday or Tuesday. (6)

Proposition (3) contains two atomic propositions viz. "I persevere" and "I will understand this". Both (4) and (5) contain the atomic proposition "Today is Monday", and (5) also contains the proposition "Today is Tuesday". Although expressed by different sentences, propositions (5) and (6) are identical and hence are treated in the same way.

You will notice that compound propositions can be expressed by beginning with sentences expressing atomic propositions and adding logical words like "if", "not", "or" etc. However it is possible to express a compound proposition by a sentence which does not contain any of these logical words. Consider the following four sentences.

"Berkeley was Irish." (7)
"Berkeley was a philosopher." (8)
"Berkeley was Irish and Berkeley was a philosopher." (9)
"Berkeley was an Irish philosopher." (10)

Clearly, sentence (10) expresses the same proposition as sentence (9), and proposition (9) is a conjunction of (7) and (8). So we may regard (10) as a compound proposition which contains (7) and (8).

Constructions like (10) need to be treated with care. As usually construed, sentence

(11) would not express a conjunction of propositions (12) and (13).

"Timothy is a big liar." (11)
Timothy is big. (12)
Timothy is a liar. (13)

For instance, Timothy might lie a lot (making (11) true) even though he is small (making (12) false.

It should now be apparent that negations, conjunctions, disjunctions, conditionals and biconditionals are all cases of compound propositions. While a propositional constant may be used to translate the whole of a compound proposition, we usually wish to translate such propositions in such a way as to show their structure in terms of the atomic components. We now investigate how this is done with the aid of our propositional operators.

Let us begin with negation. Consider the following (hopefully false) proposition:

There will be a third world war before 2000 A.D. (14)

From Chapter 1 we know that the negation of this is:

There will not be a third world war before 2000 A.D. (15)

If we represent (14) by the propositional constant "W", then (15) may be conveniently abbreviated as "*not* W". We may now construct a truth table for the negation of W as shown.

W	*not* W
1	0
0	1

The first row of this table relates to the case (more strictly, the set of all those *possible worlds*) where W is true: in any such world *not* W will obviously be false. The second row considers the case where W is false: in any such world *not* W will be true.

It doesn't really matter what particular proposition we choose. The negation truth table will always fall into the following pattern:

p	*not* p
1	0
0	1

Does this remind you of one of our propositional operators? It should! This is precisely the way $\sim$ was defined.

p	$\sim p$
1	0
0	1

So $\sim$ *corresponds precisely to negation. We may use "* $\sim$ *" to translate "not"* or, more strictly, *"it is not the case that"*. Thus (15) may be translated as "$\sim W$".

Now let us turn to conjunction. You will remember from §1.4 that proposition (18) is a conjunction of (16) and (17).

Jane studies maths. (16)
Jane studies logic. (17)
Jane studies maths and logic. (18)

If we denote proposition (16) by "M" and (17) by "L" then (18) may be written as "M

and L". From the first chart in § 1.4 the following truth table may be constructed.

M	*L*	*M and L*
1	1	1
1	0	0
0	1	0
0	0	0

M and L will be true iff both *M* and *L* are true. Provided "*and*" is used in the sense of conjunction (i.e. it simply connects two individual propositions which are both asserted to be true) we may provide a general truth table for conjunction as follows:

p	*q*	*p and q*
1	1	1
1	0	0
0	1	0
0	0	0

Which operator does this remind you of? You're right of course! This is precisely the way we defined &.

p	*q*	*p* & *q*
1	1	1
1	0	0
0	1	0
0	0	0

So & *corresponds to conjunction. We may use "&" to translate conjunctive uses of "and"* (or any other equivalent expression). This should be easy to remember, since "&" is often used in everyday English with the sense of "and" (although as a connective between nouns rather than propositions e.g., "Cobb & Co."). Thus (18) may be translated as "*M* & *L*".

Besides "and", phrases like "but" and "although" are often translated in PC by "&". A discussion of these and various non-conjunctive uses of such expressions will be given in §2.6.

Next on the agenda are disjunctions. Consider the following sentence.

"Tom studies either maths or logic." (19)

In the absence of information to the contrary, we should not rule out the possibility that Tom studies both maths and logic. So we will interpret (19) in the same way as:

"Tom studies either maths or logic, or both." (20)

Adopting the following dictionary

M = Tom studies maths
L = Tom studies logic

and treating "*or*" in the sense of *inclusive* disjunction (i.e. we allow that both disjuncts might be true), we may write (19) as "*M or L*". From the second chart in § 1.4 the following truth table may be constructed.

M	*L*	*M or L*
1	1	1
1	0	1
0	1	1
0	0	0

M or L will be true iff at least one of M and L are true. Provided "*or*" is read in the sense of inclusive disjunction, we may write a general truth table for inclusive disjunction as follows:

p	q	p *or* q
1	1	1
1	0	1
0	1	1
0	0	0

Which operator does this remind you of? You're right again! This is precisely the way we defined $\vee$.

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

So $\vee$ *corresponds to inclusive disjunction. We may use "$\vee$" to translate the inclusive "or"* (or any equivalent expression). Thus (19) may be translated as "$M \vee L$". Another phrase sometimes used to express inclusive disjunction is "at least one of".

Now consider the following *exclusive* disjunction.

Sue studies maths or logic, but not both. (21)

If we adopt the following dictionary

M = Sue studies maths
L = Sue studies logic

then the third chart in § 1.4 indicates the following truth table.

M	L	*M or L but not both*
1	1	0
1	0	1
0	1	1
0	0	0

M or L but not both will be true iff exactly one of M and L is true. The general truth table for exclusive disjunction may be written thus:

p	q	*p or q but not both*
1	1	0
1	0	1
0	1	1
0	0	0

Which operator does this bring to mind? As the table below shows, this is precisely the way we defined $\not\equiv$.

p	q	$p \not\equiv q$
1	1	0
1	0	1
0	1	1
0	0	0

So $\not\equiv$ *corresponds to exclusive disjunction. We may use "$\not\equiv$" to translate "... or ... but not both"* (or any equivalent expression). Thus (21) may be translated as "$M \not\equiv L$".

Other phrases sometimes used to express exclusive disjunction include "or" (exclusive), "exactly one of" and "has the opposite truth value to".

As mentioned in §1.4, it is not always clear which type of disjunction (inclusive or exclusive) the English "or" is being used to express. This uncertainty is indicated by the "1 or 0" in row one of the following table.

p	q	p *or* q	$p \vee q$	$p \not\equiv q$
1	1	1 or 0	1	0
1	0	1	1	1
0	1	1	1	1
0	0	0	0	0

Notice that whether p *or* q is inclusive or exclusive, whenever its value is true so is the value of $p \vee q$. We express this result by saying that p *or* q *implies* $p \vee q$, or more briefly as "*or implies* $\vee$". The possibility that "or" may be used inclusively means that "or implies $\not\equiv$ " will NOT be a general truth.

The above table also reveals that whether p *or* q is inclusive or exclusive, whenever $p \not\equiv q$ is true so is p *or* q. That is, $p \not\equiv q$ *implies* p *or* q. In brief, "$\not\equiv$ *implies or*". It is NOT generally true however that "$\vee$ implies or" (Why?).

Recall from §1.4 the standard ways of negating conjunctions and disjunctions. To negate the conjunction "Sue studies both maths and logic" we may say

It's not the case that Sue studies both maths and logic. (22)

or equivalently,

Sue doesn't study maths or she doesn't study logic. (23)

These may be translated as follows.

$\sim(M \,\&\, L)$
$\sim M \vee \sim L$

To negate the inclusive disjunction "Sue studies either maths or logic" we could say

It's not the case that Sue studies either maths or logic (24)

or equivalently,

Sue studies neither maths nor logic. (25)

Both (24) and (25) may be translated as

$\sim(M \vee L)$

Another way of saying this is

Sue doesn't study maths and she doesn't study logic. (26)

This may be translated as

$\sim M \,\&\, \sim L$

In general, a conjunction $p \,\&\, q$ may be negated by $\sim(p \,\&\, q)$ or by $\sim p \vee \sim q$. A disjunction $p \vee q$ may be negated by $\sim(p \vee q)$ or by $\sim p \,\&\, \sim q$.

You have now seen how four of our six PL operators may be used in translation. A more detailed discussion on this will be given in §2.6. Before investigating the other two operators, you should work through the following exercise. Remember that $\sim$ *corres-*

ponds to negation, & to conjunction, $\vee$ *to inclusive disjunction and* $\not\equiv$ *to exclusive disjunction.* As a general rule, *translate "or" by* "$\vee$" instead of "$\not\equiv$" *unless you are certain the exclusive sense is intended.* In Ch. 7 this simple rule will be replaced by a more complex but more correct approach.

Note also that &, $\vee$ *and* $\not\equiv$ *are commutative* i.e. given any propositions p, q we may treat both expressions in the following three pairs as logically equivalent: $p \,\&\, q$, $q \,\&\, p$;

$$p \vee q,\ q \vee p;$$
$$p \not\equiv q,\ q \not\equiv p$$

That this is so may be seen from the truth table definitions or by realizing that "*p and q*" means the same as "*q and p*" when "*and*" is used purely conjunctively, etc. So if your translation differs from the solution in the back of the book only in the *order* of the two wffs flanking &, $\vee$ or $\not\equiv$, you may count your answer as correct.

NOTES

Our treatment of logical connectives as operators on propositions rather than sentences enables (10) to be regarded quite naturally as a conjunction of (7) and (8). Strictly speaking, conjunction is a *relative* matter: conjunctions are always conjunctions *of* certain propositions. Almost any proposition may be regarded as containing two or more assertions. For instance we might regard the proposition expressed by "My daughter Selena is sweet" as asserting both that "Selena is my daughter" and "Selena is sweet", and hence treat this proposition as a conjunction of these two assertions. If however we were not interested in separating out these individual assertions, we would tend to regard this proposition as atomic rather than compound. Again, if we are focussing our attention on the concept of illegality, we might regard the proposition "This action is illegal" as atomic; however if we intended to do something with the concept of legality, we would normally use the sentence "This action is not legal" and regard this as expressing a negation, and hence a compound proposition. Thus to some extent at least, the demarcation between atomic and compound propositions is a pragmatic one. This will present no problems in practice if we always specify our propositional dictionary.

We have introduced the term "possible world" briefly and informally, but will make further use of it. For a thorough and excellent treatment of this notion consult Bradley and Swartz's *Possible Worlds*.

In Latin a distinction was made between the two senses of "or": "*vel* ... *vel* ..." meant "either ... or ..." in the inclusive sense; "*aut* ... *aut* ... " meant "either ... or ... but not both". Our wedge symbol $\vee$ is simply the first letter of "*vel*" (it is a consonantal *u*, often written nowadays as a *v*, but pronounced as a *w*: note that in keeping with its origin the wedge should always be written *sans serif*).

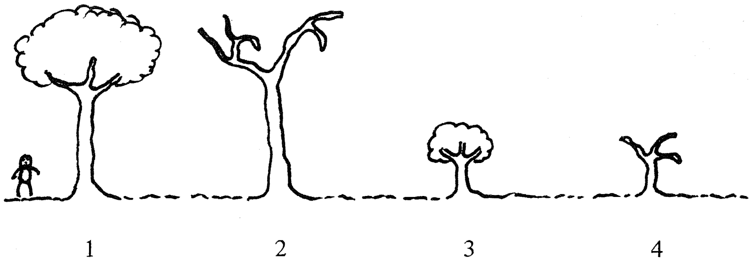
It's important to realize that $p \vee q$ does not imply that $p \,\&\, q$ is possible. $p \vee q$ just says that *at least one of p and q* is true. It is thus part of the meaning of $p \not\equiv q$. The relation between the two types of disjunction is best demonstrated by the fact that $p \not\equiv q$ is equivalent to $(p \vee q) \,\&\, {\sim}(p \,\&\, q)$. It should be clear that $p \not\equiv q$ implies $p \vee q$.

We have introduced the notion of *implication* between English and PL. This will be spelled out in greater detail later, particularly in §2.6 and Ch. 7. Our development in this area draws heavily from the pioneering work of Phillip Staines.

EXERCISE 2.4A

1.

[Question 2 asks you to translate from PL to English. Although PL is unambiguous, we have already seen that English can be ambiguous. It is important to be able to express precisely what we want to say. Question 1 is designed to help you in this regard, and to develop your sensitivity towards "*scope ambiguity*" of English words like "not" (this arises when it is unclear just how much of the rest of the sentence is modified by (i.e. in the scope of) the English word or phrase). Further examples of scope ambiguity will be met in Quantification Theory.]



For each of the following sentences, list those trees (from 1 – 4 above) for which the proposition expressed would be true. If you feel the sentence is ambiguous, state the different meanings.

(a) The tree is tall and leafy.
(b) The tree is not tall and not leafy.
(c) The tree is tall and not leafy.
(d) The tree is leafy and not tall.
(e) The tree is tall or leafy.
(f) The tree is not tall or not leafy.
(g) The tree is tall or not leafy.
(h) The tree is leafy or not tall.
(i) The tree is not tall and leafy.
(j) The tree is not both tall and leafy.
(k) The tree is both not tall and leafy.
(l) The tree is not tall or leafy.
(m) The tree is either not tall or leafy.
(n) The tree is not either tall or leafy.
(o) The tree is neither tall nor leafy.
(p) Both of these trees are leafy.
(q) It's not the case that both of these trees are leafy.
(r) Both of these trees are not leafy.

2. Use the dictionary supplied to translate the following formulae into English.

T = The tree is tall
L = The tree is leafy
B = The tree is beautiful

(a) T
(b) $\sim T$
(c) $L \& B$
(d) $T \vee L$
(e) $T \not\equiv L$
(f) $(L \& B) \& \sim T$
(g) $\sim T \& \sim L$
(h) $\sim(T \& L)$
(i) $\sim(T \vee L) \& B$

3. Use the dictionary supplied to translate the following sentences into PL.

S = Linda is a student
P = Linda is pretty
G = Linda is a girl
T = Tom is a student
B = Tom is a boy

(a) Linda is a pretty girl.
(b) Tom is not a student.
(c) Either Linda is a girl or Tom is a boy.
(d) It's not true that Tom is not a boy.
(e) Tom is a boy student and Linda is a girl.
(f) Both Linda and Tom are students.
(g) Neither Linda nor Tom are students.
(h) It's not the case that both Linda and Tom are students.
(i) Either Linda is not a student or Tom is not a student.
(j) Of Tom and Linda, just one is a student.

Propositional Calculus is also known as "*Truth Functional Logic*" since the truth value of any proposition expressed in PL is a *function* of the truth values of its atomic components. That is, for any particular assignment of truth values to the atomic components there is a *unique* truth value for the whole proposition: this is evident from the truth-tabular definitions of the PL operators. For example, given any proposition p which = 0 and any proposition q which = 1, then p & q must have the value 0.

Now just because propositional operators in PL have been deliberately defined to be truth functional, this does not mean that propositional operators expressed in a natural language like English have to be truth functional. We have already hinted that the operator "and" has non-conjunctive uses, and noted that "or" has at least two distinct roles (one inclusive, another exclusive). So as we now turn to investigate the English conditional operator "if ... then ... " we should not be unduly surprised if we find that it fails to be truth-functional.

Propositions of the form "if p then q" often crop up in arguments, so it is important to obtain a suitable translation for this expression. Unfortunately the English phrase "if ... then ... " is so ambiguous truth-functionally that no precision translation is possible. The following examples should make this clear.

		p	q	If p then q
1.	If $5 > 3$ then $5 > 2$	1	1	1
2.	If $5 > 3$ then Einstein was a scientist.	1	1	0
3.	If $5 > 3$ then $5 < 2$.	1	0	0
4.	If $5 < 2$ then $5 < 6$.	0	1	1
5.	If $5 < 2$ then $5 > 2$.	0	1	0
6.	If $5 < 2$ then $5 < 3$.	0	0	1
7.	If $5 < 2$ then kangaroos drive cars.	0	0	0

(You may feel uneasy about our "natural language evaluation" of rows 2, 5 or 7. No matter: if you disagree, this merely demonstrates the ambiguity of the expression in another way.)

We may summarise these cases in a truth table:

p	q	If p then q
1	1	1 or 0
1	0	0
0	1	1 or 0
0	0	1 or 0

Quite clearly, the English phrase "if ... then ... " is NOT truth functional. For example, given that $p = 0$ and that $q = 1$, we cannot say that *If p then q* must have the value 1, or that it must have the value 0: it could be either, depending on the particular propositions that we substitute for p and q. Since all PL operators have been defined to be truth functional, there is no single PL operator which corresponds precisely to "if ... then ... ".

However, because of the prominent role of "if ... then ... " in logical reasoning we need to find at least a good approximation for it. Fortunately, one of our PL operators will prove satisfactory for most of our purposes. It is $\supset$, and we place its truth table beside that of "if ... then ... " for comparison.

p	q	$p \supset q$	*If p then q*
1	1	1	1 or 0
1	0	0	0
0	1	1	1 or 0
0	0	1	1 or 0

Given that we want a truth-functionally unambiguous approximation for *If p then q*, then $p \supset q$ is our best available. We can demonstrate this by means of the above comparison table. The first thing to note is that $p \supset q$ agrees with *If p then q* on the second row. This is crucial: if p is true and q is false it is always a mistake to assert that if p is true then so is q. For example, the following proposition is quite definitely false:

If cats are animals then cats are plants. (27)

It is also clear from the table that whenever *if p then q* is true so is $p \supset q$ (consider rows 1, 3, and 4). We may express this fact by saying that *if... then... implies* $\supset$. Because of this implication there are important logical features of *if. . . .then. . .* that are possessed by $\supset$. As we will see later, the three most important valid argument–forms involving *if. . . then. . .* (AA, DC and Ch Ar) are parallelled by valid argument–forms involving $\supset$.

The two requirements of agreement on row two, and implication by "if ... then ... ", uniquely determine $p \supset q$ as our translation for *If p then q*. There are other motivations for choosing $\supset$ (e.g., our translation must have the value 1 on rows one and four, otherwise expressions of the form *If p then p* would be translated as false). Although "if ... then ... implies $\supset$" it is clear from the above table that $\supset$ does NOT imply *if ... then ...* (Why?).

Logicians sometimes refer to the $\supset$ operation as "material implication". Thus "$p \supset q$" is sometimes read as "*p materially implies q*". This is motivated by the knowledge that if $p \supset q$ is true and p is true then as a matter of fact q will have to be true too. However the connection between p and q which is expressed in English by "If p then q" is much stronger than mere material implication. This point will be taken up in §2.6, but we note here two important differences between $\supset$ and *if ... then ...* :

1. $\sim p \supset (p \supset q)$

   If p is false then $p \supset q$ is true: a false proposition materially implies any proposition (this follows from rows 3 and 4 of the table for $\supset$ e.g.,

   The Earth has two moons materially implies that
   the Earth has ten moons. (28)

2. $q \supset (p \supset q)$

   If q is true then $p \supset q$ is true: a true proposition is materially implied by any proposition (this follows from rows 1 and 3) e.g.,

   Today is Friday materially implies that the Earth has one moon (29)

These two facts about $\supset$ are examples of the so-called *"paradoxes of material implication"*. They appear paradoxical only if we blindly imagine that $\supset$ captures precisely the meaning of "if ... then ... ". Despite these notorious failures of $\supset$ to match exactly our intuitive sense of "if ... then ... ", we will often use it to translate conditionals into PL. The important thing to remember is that "if ... then ... implies $\supset$". In Chapter 7 we will look again at this problem and indicate the general circumstances under which translation by $\supset$ is justified. For the present however, we will assume that $\supset$ may be used to translate *if ... then ...* into PL.

We saw earlier that &, $\vee$ and $\not\equiv$ are commutative. In Ex. 2.3 you established that $p \supset q$ has a different truth table from $q \supset p$. So $\supset$ *is not commutative* i.e. the order of the operands around the $\supset$ does matter. In §1.5 we saw that *if p then q* is not equivalent to its converse *If q then p*. So $\supset$ correctly reflects the non-commutativity of conditionals. Because of this we often describe $q \supset p$ as the converse of $p \supset q$.

You will recall from §1.5 that the conditional *If p then q* may also be expressed as *p only if q,* and that its converse *If q then p* may also be expressed as *p if q*. Now look at the truth table below.

p	q	$p \equiv q$	$p \supset q$	$q \supset p$	$(p \supset q)$ & $(q \supset p)$
1	1	1	1	1	1
1	0	0	0	1	0
0	1	0	1	0	0
0	0	1	1	1	1

Here we have used a common matrix to serve for four different formulae. The expression $(p \supset q\ )$ & $(q \supset p)$, being a conjunction, will be true iff both of its conjuncts (viz., $p \supset q$, $q \supset p$) are true; as can be seen from the columns for these conjuncts, this happens on rows 1 and 4 but not on rows 2 and 3. Hence the column for $(p \supset q)$ & $(q \supset p)$ is as shown; but this is precisely the same as the column for $p \equiv q$. Therefore *$p \equiv q$ is equivalent to $(p \supset q)$ & $(q \supset p)$*. Thus "$p \equiv q$" may be used to translate "If p then q, and if q then p". Using the alternative readings mentioned above this becomes "p only if q, and p if q" or more neatly "p if and only if q". Hence *"$p \equiv q$" may be used to translate "p iff q".*

So we now have a way of translating *biconditionals* in PL. If we adopt the following dictionary

E = The set is empty
N = The set has no members

then we will translate the biconditional

The set is empty if and only if it has no members. (30)

by

$E \equiv N$

But how satisfactorily does $\equiv$ capture the sense of *iff*? If you look at the truth table for $\equiv$ you will see that $p \equiv q$ will be true just when p and q have the same truth value. So "$p \equiv q$" may also be read "*p has the same truth value as q*". Because of this, logicians sometimes refer to the $\equiv$ operation as "material equivalence". Thus "$p \equiv q$" is sometimes read as "*p is materially equivalent to q*". But equivalence of truth value is a very weak form of equivalence. The sort of equivalence usually expressed by the phrase "if and only if" involves a much stronger connection between the operands. Any two true propositions will be materially equivalent e.g.,

Earth is inhabited is materially equivalent to $1 + 1 = 2$. (31)

Moreover, any two false propositions are materially equivalent e.g.,

The Earth has two moons is materially equivalent to
the Earth has ten moons. (32)

Clearly, it would not do to replace the connective "is materially equivalent to" in these cases with "iff".

These two examples are instances of the so-called *"paradoxes of material equivalence".* They are paradoxical only if we incorrectly assume that $\equiv$ captures precisely the meaning of "if and only if". However, just as we will mostly use $\supset$ to translate *if ... then ...* , so we will mostly use $\equiv$ to translate *iff.* Unfortunately there is no neat relation of implication between p *iff* q and $p \equiv q$. Each case will have to be judged separately (see Ch. 7). Later in this book stronger forms of implication and equivalence will be discussed which will represent more closely certain strong conditionals and biconditionals.

One further point about $\equiv$ is worth noting here. It should be obvious, either from truth tables or from reading $\equiv$ as a matching truth value operator, that the truth table for $p \equiv q$ will agree exactly with that of $q \equiv p$. So $\equiv$ *is commutative.* So of all our dyadic operators, only $\supset$ is non-commutative.

Well, we have now seen how all the basic types of propositions introduced in Chapter 1 (viz. negations, conjunctions, inclusive disjunctions, exclusive disjunctions, conditionals and biconditionals) may be translated in PL by means of propositional operators ($\sim$, &, $\vee$, $\not\equiv$, $\supset$ and $\equiv$ respectively).

Having discovered basic translation uses for each of our operators, it would be helpful to practise these uses before considering trickier cases in the next section. With translation exercises, any answer logically equivalent to the provided answer will be correct. Logical equivalence will be discussed in depth later, but we note here two ways in which formulae may be equivalent.

Firstly, remember that all our dyadic operators except $\supset$ are commutative. For example, the following formulae are all logically equivalent:

$(p \,\&\, q) \vee (r \equiv s)$
$(q \,\&\, p) \vee (s \equiv r)$
$(s \equiv r) \vee (q \,\&\, p)$

Secondly, remember that we are free to use different types of brackets and to drop brackets when no ambiguity results. Apart from deleting outermost brackets, further freedom with brackets follows from the fact that some of our dyadic operators are *associative.* Associativity will be dealt with in more detail later, but we note here that &, $\vee$, $\equiv$, $\not\equiv$ *are associative whereas* $\supset$ *is not.* Saying that & is associative means that in any wff where & is the *only* dyadic operator it doesn't matter which conjuncts we associate first. For instance, $p \,\&\, (q \,\&\, r)$ is equivalent to $(p \,\&\, q) \,\&\, r$. Hence it is O.K. to write $p \,\&\, q \,\&\, r$ since this is unambiguous. This corresponds to the fact that with English sentences of the form *p and q and r* it doesn't change the meaning if we place a comma after p or after q: the sentence simply states that each of p, q, and r are true.

Similarly $p \vee q \vee r$, $p \equiv q \equiv r$ and $p \not\equiv q \not\equiv r$ are acceptable, but $p \supset q \supset r$ is illegal e.g., $p \supset (q \supset r)$ is quite different in meaning from $(p \supset q) \supset r$. Note in particular that brackets are always required when different dyadic operators are included in the same formula e.g., $p \,\&\, (q \vee r)$ is not equivalent to $(p \,\&\, q) \vee r$.

To assist you with the following exercise, a summary of the main translations discussed so far is now presented. Section 2.6 will consider additions to this list.

$\sim p$	not p it's not the case that p
$p \,\&\, q$	p and q both p and q p but q
$p \vee q$	p or q (inclusive) either p or q p or q or both
$p \supset q$	if p then q if p, q p only if q
$q \supset p$	p if q only if p, q
$p \equiv q$	p if and only if q if p then q, and conversely
$p \not\equiv q$	p or q but not both exactly one of p and q

Remember also that expressions of the form *Not both p and q* may be translated as $\sim(p \,\&\, q)$, and expressions of the form *Neither p nor q* may be translated as $\sim(p \vee q)$ or as $\sim p \,\&\, \sim q$.

Having done all that theory you must be anxiously awaiting some more questions to exercise your new logical muscles. You will find just what the doctor ordered in the exercise below!

NOTES

The fact that "material implication" lacks the logical force usually associated with the term "implication" has prompted some authors (e.g., Bradley and Swartz *op. cit.*) to recommend that this title be replaced by "material conditionality". Similarly it has been suggested that the term "material equivalence" be replaced by "material biconditionality". The older terms seem so entrenched in common logical usage however, that we have retained them, while warning against reading them too strongly.

In mathematics a slash is often used instead of $\sim$ to denote "not" e.g., $x \neq y$ means $\sim(x = y)$. Likewise $p \not\equiv q$ means $\sim(p \equiv q)$.

EXERCISE 2.4B

1. Translate the following PL wffs into English using the dictionary provided.

(a) F
(b) $\sim F$
(c) $F \& S$
(d) $F \vee S$
(e) $F \supset S$
(f) $F \equiv S$
(g) $W \not\equiv F$

F = Today is Friday
S = Tomorrow is Saturday
W = Today is Wednesday.

2. Given that today is Friday, which of the propositions in Question 1 would be counted as true?

3. Given that today is Monday, which of the propositions in Question 1 would be counted as true?

4. Translate the following PL wffs into English using the dictionary provided.

H = I am a human
M = I am a man
W = I am a woman

(a) $(M \vee W) \supset H$
(b) $H \equiv (M \vee W)$
(c) $\sim(M \& W)$
(d) $M \supset (H \& \sim W)$
(e) $H \supset (M \not\equiv W)$
(f) $\sim H \supset \sim(M \vee W)$
(g) $H \supset \sim(M \& W)$

5. Translate the following sentences into PL using the dictionary supplied.

I = Logic is interesting
U = Logic is useful
B = Logic is boring
M = I'm a monkey's uncle
N = You're my nephew

(a) Logic is interesting.
(b) Logic is not boring.
(c) Logic is interesting or useful.
(d) Logic is interesting and useful.
(e) Logic is interesting and useful but not boring.
(f) If logic is boring then I'm a monkey's uncle.
(g) I'm a monkey's uncle only if you're my nephew.
(h) Logic is interesting or boring but not both.
(i) I'm a monkey's uncle if and only if you're my nephew.
(j) Neither is logic boring nor are you my nephew.
(k) It's not true to say that logic is both useful and boring.

6. Translate the following sentences into PL using the dictionary supplied.

E = The number is even
O = The number is odd
P = The number is positive
Z = The number is zero
N = The number is negative

(a) The number is not even.
(b) The number is either even or not even.
(c) The number is odd only if it's not even.
(d) The number is odd if it's not even.

(e) The number is odd if and only if it's not even.
(f) If the number is negative then it's not positive.
(g) The number is not both even and odd.
(h) The number is neither positive nor negative.
(i) If the number is zero then it's neither positive nor negative.
(j) The number is positive or zero or negative.
(k) If the number is not zero then it's either positive or negative but not both.
(l) The number is not positive if it is negative.
(m) The number is even, non-zero, and positive.
(n) If the number is even then it is not odd, and conversely.
(o) If the number is either even or positive then it is not both odd and negative.
(p) The number is either even and positive, or odd and negative, but not both.
*(q) Only if the number is non-zero and non-negative will it be positive.

2.5 SENTENCES AND FORMS IN PL

Well formed formulae in PL may be conveniently divided into two main types: PL-*sentences* and PL-*forms*. For the rest of Part One we will often refer to these simply as sentences and forms.

Sentences contain at least one propositional constant, and no propositional variables. Here are some examples: "A"; "$\sim A \vee B$"; "$(A \,\&\, B) \supset C$". PL-forms on the other hand contain at least one propositional variable, and no propositional constants. Here are some examples: p; $\sim p \vee q$; $(p \,\&\, q) \supset r$. Given the dictionary of propositional constants, sentences will express definite propositions. Strictly speaking, forms never express propositions.

Given a compound proposition, there will always be more than one sentence in PL that could be used to express it. Take for instance the following proposition.

This is easy and you'll follow it. (1)

If for some reason we wanted to treat this as a unit we could specify the following dictionary

T = This is easy and you'll follow it

and translate (1) simply by the following sentence

"T"

However, if we wish to reveal the internal structure in which the atomic components reside we will choose a dictionary of atomic propositions:–

E = This is easy
F = You'll follow it

Proposition (1) may now be translated by the sentence

"$E \,\&\, F$"

We will call this sentence an *explicit sentence* of PL to indicate that all its propositional constants represent atomic propositions: it unfolds as much of the proposition's structure as is possible in PL.

When logicians want to establish general results about the logical structure of propositions and arguments, not tying themselves down to particular propositions, they deal with forms rather than sentences. The sentence "T" has just one form: p. The sentence "E &

F" also has the form p (as will any sentence), but in addition it has the form p & q. In general, a sentence *has* (is an *instance* of) a certain form iff it can be generated from that form by replacing propositional variables in that form with sentences, where all occurrences of the same variable must have similar replacements (i.e. the *substitution* must be *uniform*).

For instance, "E & F" can be generated from p by replacing "p" with "E & F". Also, "E & F" can be generated from p & q by replacing "p" with "E" and "q" with "F". Note that "E & F" does *not* have p & p as one of its forms, since both occurrences of p must be replaced by the *same* sentence for it to qualify as a form. For instance the sentence "E & E" does have p & p as one of its forms.

Let's look at another proposition.

If this is easy you'll follow it, and it is easy. (2)

Its explicit sentence will be

"$(E \supset F)$ & E"

This has four forms:

$(p \supset q)$ & p
$(p \supset q)$ & r
p & q
p

The first of these is obtained from the sentence by substituting variables for constants, introducing new variables in the alphabetic order $p, q,$ ... but using the same variables for the same constants: this form provides the maximum information on the structure of the sentence and is called its *explicit form*. Although sentences may have many forms, they will have just one explicit form.

We may now define the explicit form (in PL) of a proposition as the explicit form of its explicit sentence (in PL).

NOTES

Unlike some authors, we do not allow propositional constants to feature in PL-forms. Besides sentences and forms, a third class of PL-wffs does exist viz. *PL-hybrids*: these contain at least one variable and at least one constant e.g., $A \supset p$. Discussion of forms is simplified by excluding hybrids. We will have no use for PL-hybrids in this text.

Quotes will often be used to distinguish reference to sentences (e.g., "E & F") from reference to propositions (e.g., E & F). Quotes will usually be avoided with propositional forms.

The explicit form of a sentence is sometimes called its "specific form" or its "skeleton".

In speaking as if a proposition has a unique explicit sentence we have assumed there is a unique answer to the question as to what the proposition's atomic propositions are (see Notes to §2.4). Secondly we have ignored trivially different sentences arising from different choices of letters for the propositional constants e.g., (2) could be symbolized by "A & B" rather than "E & F". Thirdly, if it is argued that operand order around commutative operators is not a criterion for propositional identity (e.g., that E & F is the same proposition as F & E) we have ignored such differences in order. The second of these qualifications is of course not needed to establish the uniqueness of a propositional's PL-form.

Although propositions (1) and (2) can be shown to be truth-functionally equivalent (see Ch. 3), and involve the same atomic propositions, they have different explicit forms and hence are not the same proposition. (For a treatment on identity rather than mere equivalence of propositions see Bradley and Swartz *op. cit.* pp. 94-97).

EXERCISE 2.5

1. Provide explicit sentences (in PL) for the following propositions.
   (a) Romeo is happy if and only if Juliet is.
   (b) I'm not worried but I am concerned.
   (c) If it's Monday or Tuesday then it's not Wednesday.

2. For each of your answers to Question 1, write down:
   (a) the explicit form
   (b) the other forms

3. Two certain propositions have the same explicit form. Must they be the same proposition? Give an example to back up your answer.

2.6 TRANSLATION: SYNONYMY, EQUIVALENCE AND IMPLICATION

When English sentences are translated into PL-sentences, two conditions should ideally be satisfied. Firstly, the PL-sentence should *express the same proposition* as the English sentence i.e. the two sentences should be *synonymous*. Secondly, the PL-sentence should *display the atomic propositions involved and their attendant logical operators* (except when, for reasons of efficiency, compound propositions are treated as a unit). By "attendant" logical operators we mean those operating on the atomic propositions.

In practice these two requirements for an exact translation are rarely met. In the first place, it can sometimes prove awkward to select a dictionary of propositional constants that exactly matches all the atomic propositions involved. More importantly, the operators of PL, defined in §2.3 by means of truth tables, often fail to capture the precise meaning and nuances of the English operators they are used to translate: the most notorious failure in this regard is the use of "$\supset$" as a translation for "if ... then ... ", as discussed in §2.4. Despite these problems, a great deal can be accomplished. By becoming acquainted with the types of problems that can arise and methods for overcoming them, we will be able to make effective use of PL for analysing propositions and arguments.

To appreciate that a problem can arise when choosing a dictionary of atomic propositions, let's consider the following valid argument.

If love is alive then there's hope for the world.
Obviously, love is alive.
So there's hope for the world. (1)

Strictly speaking there are three different atomic propositions here, which we might symbolize as follows:

L = Love is alive
H = There is hope for the world
O = Obviously love is alive

That L differs from O should be clear if we compare their negations:

Love is not alive. (2)
It's not obvious that love is alive. (3)

It should be clear that (2) and (3) are quite different, and that (4) is different again.

Obviously love is not alive. (4)

Argument (1) may thus be abbreviated as follows:

$$\begin{array}{l} \text{If } L \text{ then } H \\ \quad O \\ \hline \therefore \quad H \end{array} \tag{1a}$$

This has the following form:

$$\begin{array}{l} \text{If } p \text{ then } q \\ \quad r \\ \hline \therefore \quad q \end{array}$$

Now since not all arguments of this form will be valid, translating (1) as (1a) fails to expose the valid structure of (1). What needs to be seen is the logical connection between L and O. This can be made clear by using the notions of *equivalence* and *implication.* We have met these notions before, and will have a lot more to say about them in future chapters, but for the moment let us agree that *p is equivalent to q* iff *p and q must always have the same truth value,* and that *p implies q* iff *whenever p is true so is q.*

If you think about it you will see that O implies L. Another way of expressing this is to say that L is an *implicant* of O. Moreover, O has identical truth conditions to the conjunction *O and L*: whenever O is true so is *O and L*; and vice versa. So O is equivalent to *O and L*. In general, any proposition will be equivalent to a conjunction of itself and an implicant (this conjunction is sometimes called a "conjoint product" of the original proposition).

Since the logical classification of propositions and propositional relations (including validity of arguments) depends only on truth conditions, it is permissible for purposes of such classification to translate sentences of English into equivalent sentences of PL. Such translations may be called *equivalent translations*. Some but not all equivalent translations will be synonymous translations.

Any translation which is either equivalent or synonymous will be called an *accurate translation.*

Here is one equivalent translation for argument (1):

$$\begin{array}{l} \text{If } L \text{ then } H \\ \quad O \text{ and } L \\ \hline \therefore \quad H \end{array} \tag{1b}$$

Now the valid structure of (1) is apparent, because *O and L* implies L, and L combined with the first premise yields the conclusion.

Quite often, logicians adopt a simpler approach still in translating propositions for logical analysis. Rather than translating from English to a synonymous or equivalent PL-sentence, they sometimes translate to a weaker PL-sentence which is implied by the English, provided this is adequate for their purposes. Such translations may be called *implied translations.* For instance since O implies L, and L is sufficient to establish validity, O may be translated simply as L for purposes of assessing argument (1). Thus (1) may be presented as:

$$\begin{array}{l} \text{If } L \text{ then } H \\ \quad L \\ \hline \therefore \quad H \end{array} \tag{1c}$$

which is obviously valid.

Having considered the translation of atomic propositions, let's have a look now at the translation of logical operators. In most cases when English operators are translated into PL-operators, the translation will be equivalent or implied rather than synonymous. With negation however, exact translations are usually found. For instance, given the dictionary

R = It is raining

each of (5) and (6) is synonymous with (7).

It's not raining. (5)
It's false that it's raining. (6)
$\sim R$ (7)

In some cases it is convenient to translate phrases like "can't" and "impossible", which imply negation, simply in terms of "$\sim$". For example argument (8) is more easily seen to be valid from the implied translation (8a), using the dictionary:

C = His theory is correct
P = Computers feel pain

If his theory is correct then computers feel pain.
It's impossible for computers to feel pain.
Hence his theory is incorrect. (8)

$$\begin{array}{l} \text{If } C \text{ then } P \\ \sim P \\ \hline \therefore\ \sim C \end{array} \qquad (8a)$$

However, as we will see in Chapter 3, there are cases where a sharp distinction needs to be drawn between "not" and "not possible".

Now let's look at conjunctions and related cases. When translating with "&", synonymy or at least equivalence will usually be attainable. Given the dictionary

H = Hindus believe in a God
M = Muslims believe in a God

the conjunction (9) translates exactly as (10).

Hindus believe in a God and Muslims do too. (9)

H & M (10)

The next example is more than a simple conjunction, because it contains an element of contrast.

It's humble, but it's my home. (11)

The atomic propositions here may be symbolized:

H = It's humble
M = It's my home

"But" is often used to "discount" the proposition before it in favour of the following proposition. Notice the difference between (11) and (12).

It's my home, but it's humble. (12)

Despite these differences, both (11) and (12) are truth-functionally equivalent to (13): each is true iff both H and M are true. So for purposes of logical analysis (11) would usually be bluntly translated as (13).

H & M (13)

Similarly, the subtler aspects of many other English conjunctives will usually be ignored in logical translation (unless these aspects have a bearing on the logical task at hand). For instance, given the dictionary

B = She's the firm's best engineer
W = She's a woman

the proposition

She's the firm's best engineer even though she's a woman. (14)

would normally be translated as

$B \& W$

even though the connective "even though" goes beyond the meaning of "&" in suggesting that it is unusual for the best engineer to be a woman.

A word or two needs to be said about the translation of "because" or "since". If *p because q* is merely one premise in an argument, then it is often acceptable to translate it simply as $p \& q$. You might feel that $q \& (q \supset p)$ would be more correct, but it is easy to show by truth tables that this is equivalent to $p \& q$ (you will learn how to do this in Ch. 3). If however, p is the conclusion of the argument we should instead display *p because q* as

$$\frac{q}{\therefore p}$$

A similar comment holds for propositions of the form *p since q*. Note that the "___" and " ∴ " are not symbols of PL but they are part of the symbolic notation adopted within PC for the analysis of arguments.

As indicated in the notes to §2.4, it is sometimes useful to treat sentences with one verb as expressing conjunctions, if the separate assertions can be made explicit by paraphrasing. For example (15) and (17) might be rephrased as (16) and (18) respectively.

It's raining in spite of the sun's shining. (15)
It's raining and the sun is shining. (16)
Looking through a telescope, Halley saw a comet. (17)
Halley looked through a telescope and saw a comet. (18)

Quite often the word "and" is used to convey more than conjunction. In (19) it has a *temporal* sense indicating he ate the pizza before he got the tummy ache.

He ate a pickled pizza and got a tummy ache. (19)

In §2.4 we noted that & is commutative i.e. in all cases $p \& q$ is equivalent to $q \& p$. That "and" is not commutative can be seen by comparing sentences (19) and (20).

He got a tummy ache and ate a pickled pizza. (20)

If you try swapping the atomic propositions in (18) you will notice that there too "and" has the sense of "and then". In such cases "and" will commonly be translated as "&" unless the temporal aspect is logically important.

Not every use of "and" will imply &. Consider the following proposition.

You do me a good turn and I'll do you a good turn. (21)

Here "and" acts as a conditional rather than a conjunctive operator. The same proposition could be expressed as:

If you do me a good turn then I'll do you a good turn. (22)

Rather than mechanically replacing words with symbols, a little common sense is needed to ensure that we translate what is *expressed* by the English.

Care should also be taken when negation and conjunction are combined. Given the dictionary

E = The Earth is a planet
S = The Sun is a planet

how would you translate the following propositions?

The Earth and the Sun are not both planets. (23)
Both the Earth and the Sun are not planets. (24)

It should be clear that (23) and (24) are not equivalent. Proposition (23) allows that one of the Earth and the Sun may be a planet: it simply denies that both of them are planets, and hence may be translated as the negated conjunction (25).

$$\sim(E \,\&\, S) \tag{25}$$

On the other hand, the sentence (24) would normally be construed as saying that neither the Earth nor the Sun is a planet: hence it may be treated as a conjunction of negations, as follows.

$$\sim E \,\&\, \sim S \tag{26}$$

On this account, (23) is a true proposition and (24) is false, since we know that the Earth, but not the Sun, is a planet. Notice that the sentences (23) and (24) are identical except for the position of "both". It is not uncommon for a change in word position to lead to a change in meaning: part of our job as logicians is to make ourselves more sensitive to such differences.

Earlier we saw that "not" could sometimes be used as an implied translation for "impossible". To say that two propositions are incompatible is to say that it is impossible for both of them to be true (though one might be true): this implies that they are not both true. Consider the following example.

The Earth's being a planet is incompatible with the Sun being a planet. (27)

This implies that the Earth and the Sun are not both planets. So in some cases (25) might be used as an implied translation for (27). An exact treatment of the notion of incompatibility (or inconsistency) will be given in Chapter 3.

Let's spend a minute or two now on disjunctions. Recall that inclusive disjunction is handled by $\vee$, and exclusive by $\not\equiv$. With expressions of the form "*p* or *q* or both" or the legal "*p* and/or *q*", the disjunction is clearly inclusive. Some uses of "or" are clearly inclusive and some are clearly exclusive: when in doubt we treat it as inclusive (a more sophisticated approach will be given in Ch. 7).

In some sentences, "or" is used with the sense of "that is" e.g.,

Incompatibility, or inconsistency, will be examined later. (28)

Manifestly, *or implies* $\vee$ here also, so $\vee$ could be used in an implied translation: much later in the book the identity relation "=" will allow an exact treatment of such cases.

An important logical phrase not yet considered is "unless". Consider the following proposition.

Your diet will be useless unless you exercise. (29)

This implies, and is perhaps equivalent to:

If you don't exercise, your diet will be useless. (30)

This seems equivalent to:

Either you exercise, or your diet will be useless. (31)

Choosing the dictionary

D = Your diet will be useless
E = You exercise

and for safety-sake treating the "or" in sentence (31) as inclusive, (31) may be translated as:

$E \vee D$

Since $\vee$ is commutative, this is equivalent to:

$D \vee E$

By virtue of the linkage between (29) and (31), which is at least as strong as implication, this will usually be taken as an acceptable translation for (29). In other words, "D unless E" translates as "$D \vee E$". Sometimes "unless" is placed at the front instead of between e.g.,

Unless you exercise, your diet will be useless. (32)

This is equivalent to (29) and hence may be translated in the same way. In general, our normal practice will be to translate expressions of the form *p unless q* or *Unless q, p* simply as $p \vee q$. Because $\vee$ commutes, it is also usually permissible to translate *Unless p, q* simply as $p \vee q$.

As a final note on $\vee$, recall from Chapter 1 that expressions of the form *neither p nor q* are equivalent to negated inclusive disjunction and so may be translated as $\sim(p \vee q)$ or the equivalent $\sim p \mathbin{\&} \sim q$.

Exclusive disjunctions of the form *p or q but not both* may be exactly translated by $p \not\equiv q$. When just two alternatives are involved, the phrase "just one of" may also be handled by $\not\equiv$. For example, given an obvious dictionary, (33) translates as (34).

Just one of John and Bill is a cricketer. (33)

$J \not\equiv B$ (34)

Unfortunately, when more than two alternatives are involved it gets more complicated. For instance (35) is *not* equivalent to (36)

Just one of John, Bill and Tom is a cricketer. (35)

$(J \not\equiv B) \not\equiv T$ (36)

(36) is true not only when just one of J, B and T is true but also when all three are true (this may be shown with the aid of a truth table). So (35) may be translated by (37).

$[(J \not\equiv B) \not\equiv T] \mathbin{\&} \sim(J \mathbin{\&} B \mathbin{\&} T)$ (37)

Let's turn now to conditionals and biconditionals. From earlier work (§1.5), we can say that each of (38) to (41) may be treated as the same conditional. By comparison with sentence (41) it is clear that sentence (42) also expresses the same conditional.

If Karen is at home *then* the stereo is turned up. (38)
If Karen is at home the stereo is turned up. (39)
The stereo is turned up *if* Karen is at home. (40)
Karen is at home *only if* the stereo is turned up. (41)
Only if the stereo is turned up is Karen at home. (42)

The atomic propositions involved may be symbolized as:

K = Karen is at home
S = The stereo is turned up

In this conditional K is the antecedent and S is the consequent. Notice that when used, *"if" precedes the antecedent* and *"only if" precedes the consequent.* This is generally true. As speech acts, the "if" form of the conditional might be contrasted with the "only if" form as follows: the "if" form emphasizes that if the antecedent is true then so is the consequent; the "only if" form emphasizes that if the consequent is false then so is the antecedent. This serves to remind us that *true* conditionals have these two features: a true antecedent yields a true consequent, and a false consequent yields a false antecedent.

Recalling from §2.4 that "if ... then ... " implies $\supset$ and that $\supset$ is the PL-operator nearest in meaning to "if ... then ... ", each of (38) to (42) may be given the implied translation

$$K \supset S \tag{43}$$

Conditionals are sometimes expressed using "whenever" or "when". For example, both (44) and (45) may be translated as (43).

Whenever Karen is at home the stereo is turned up. (44)
When Karen is at home the stereo is turned up. (45)

Care must be taken though with "when". If the atomic sentence it precedes refers to a specific past or future event it will rarely mean "whenever". Consider the following case.

When I was at the bank I deposited my pay cheque. (46)

Here "When" means "On the previous occasion" : it conveys both conjunction and temporal coincidence. The use of "when" in relation to future events usually indicates the speaker's belief that the event will occur. Compare the following:

When I pass the exam, we'll celebrate. (47)
If I pass the exam, we'll celebrate. (48)

This difference is sometimes emphasized by the phrase "Not 'if', but 'when' ". In cases like this, "when" is better translated by & than by $\supset$, but the temporal connection is still lost.

A proposition of the form "If p then q" may sometimes be expressed by saying that p is a *sufficient* condition for q (since given p, q follows), or by saying that q is a *necessary* condition for p (since you can't have p without also having q). As with "if ... then ... ", the best we can do in PL is use $\supset$ to provide an implied translation. For example both (49) and (50) may be translated by sentence (43).

Karen's being at home is sufficient for the stereo to be turned up. (49)
The stereo being turned up is a necessary condition for Karen's being at home. (50)

Notice how noun phrases or "nominalizations" are used instead of full sentences here to express the actual conditions. This is fairly common.

Since "p if q" means "p is necessary for q", and "p only if q" means "p is sufficient for q", it follows that "p iff q" may be rephrased as "p is necessary and sufficient for q". Just as we use $\equiv$ as an implied translation for "iff" then, we may use it to translate "is necessary and sufficient for".

That's enough for now on translating English phrases into PL-operators. You will find a list of the translations we have discussed (plus a few others) in the summary of §2.8. You may find this list helpful as a translation aid, but don't use it mechanically: it is simply a general guide which may be overriden when we detect that the English phrase is being used in a different sense.

Bracketing:

Before getting your teeth into the translation exercise that follows, here are two strategies that might assist you with bracketing: these may be called the *top-down* method and the *bottom-up* method.

Let's consider the top-down method first, and use the following proposition as an example.

> If the Prime Minister supports the policy and the Cabinet does not, then the Prime Minister will either give way or resign. (51)

First we detect the main operator and symbolize it. We ask, "What is the proposition *as a whole*?" Our example is a conditional. So we symbolize the main operator and leave the rest as it is, to produce the following hybrid (i.e. combination) of English and PL:

> The Prime Minister supports the policy and the Cabinet does not $\supset$ the Prime Minister will either give way or resign (52)

Since outer brackets are redundant it is not necessary to enclose this hybrid with brackets; if the main operator was $\sim$ however and its negand was a compound proposition, it would be necessary to enclose this negand in brackets. From this stage onwards, every time we symbolize with a dyadic operator of PL or with a $\sim$ that has scope over a compound proposition it will be necessary to insert brackets. With this in mind, we now look at the untranslated parts, symbolize the "main operator" of each part, and continue this procedure until the only untranslated parts are atomic propositions. Looking at the antecedent of (52) we see that it is a conjunction so we symbolize to get:

> (The Prime Minister supports the policy & the Cabinet does not) $\supset$ the Prime Minister will either give way or resign (53)

Then the consequent, which is a disjunction, is symbolized:

> (The Prime Minister supports the policy & the Cabinet does not) $\supset$ (the Prime Minister will give way $\vee$ the Prime Minister will resign) (54)

Given our preference for atomic propositions that are affirmative, the second conjunct of the antecedent will be treated as a negation and hence symbolized as follows:

> (The Prime Minister supports the policy & $\sim$ the Cabinet supports the policy) $\supset$ (the Prime Minister will give way $\vee$ the Prime Minister will resign) (55)

We can now set out a dictionary for the four atomic propositions:

S = The Prime Minister supports the policy
C = The Cabinet supports the policy
G = The Prime Minister will give way
R = The Prime Minister will resign

Using this dictionary we get:

$$(S \mathbin{\&} \sim C) \supset (G \vee R) \qquad (56)$$

The top-down method not only helps us to see the logical structure of a proposition, but also gives us a dictionary. However in some cases, especially in logic text books and examinations, a dictionary is provided. For example we might be given the following dictionary and be asked to symbolize the proposition set out under it.

N = There will be a nuclear accident
M = Many lives will be lost
R = Money will be spent on research
W = A safe waste disposal system will be discovered.

Either there will be a nuclear accident and many lives will be lost or both money will be spent on research and a safe waste disposal system will be discovered. (57)

In a case like this, the bottom-up method can be used. First we use the propositional constants to get:

Either N and M or both R and W (58)

This then leads to

Either $(N \& M)$ or $(R \& W)$ (59)

and finally

$$(N \& M) \vee (R \& W) \tag{60}$$

In practice, either the top-down or the bottom-up method may be used with any translation. You may like to combine the methods. One highly recommended technique is to first establish your dictionary of propositional constants (underlining parts of the original English sentence can help you find the atomic propositions), substitute these in, and then work top-down to provide the operators and brackets. The emphasis in the top-down method of searching for the main-operator is very helpful.

Notice how the use of "either" and "both" in (58) disambiguates the sentence by fixing where the brackets must be placed. If we leave them out we get:

N and M or R and W (61)

This is highly ambiguous. Can you spot five different ways of bracketing (61)? Links between words like "either" and "or", "both" and "and", and "if" and "then" assist us a great deal in deciding where to place brackets. See how you go with inserting brackets in the following example: use the letters "C", "L", "H", "M" and "D" for your dictionary.

If either both Confucius and Lao Tzu were alive or both Hobbes and Mill then there would be an amazing debate. (62)

If you haven't already done so, complete the translation with the aid of PL-operators and then check your answer with (63).

$$[\,(C \& L) \vee (H \& M)\,] \supset D \tag{63}$$

Don't forget that commas often give a clue as to where brackets should be inserted. Remember these two examples from §2.3?

Earth is a star and Venus is a star, or Sirius is a star. (64)

Earth is a star, and Venus is a star or Sirius is a star. (65)

Using the letters "E", "V" and "S" for our dictionary, (64) translates as

$$(E \& V) \vee S \tag{66}$$

While the sentence (65) is mildly ambiguous, it would be usual to translate it as expressing the following proposition:

$$E \,\&\, (V \vee S) \tag{67}$$

Well, off you go now (at last!) to try out your new logical muscles on the following exercise. Don't forget the translation guide in the next section is there to help you.

NOTES

A useful discussion of equivalent and implied translations is contained in Appendices C, E and F of *Elementary Applied Symbolic Logic* by B. L. Tapscott. We will treat the notion of implied translation at greater depth in Ch. 7.

If you look at sentence (62), the use of "were" and "would" instead of "are" and "will" indicates the conditional is cast in the subjunctive mood. ***Subjunctive conditionals*** are often "***counterfactual***" i.e. they are understood to imply that the antecedent is counter to fact (i.e. false). Clearly, (62) is a counterfactual conditional. Some subjunctive conditionals are not counterfactual conditional e.g., "If I were to win the lottery I'd go on a world trip": this leaves the question open as to winning the lottery. Subjunctive conditionals usually *imply* their indicative counterparts, which in turn imply $\supset$ conditionals, so in some cases we may translate subjunctive conditionals in terms of $\supset$ (we did this when we translated (62) into (63)). Be careful not to assume $\supset$ conditionals are *equivalent* to counterfactuals however, because then *all* counterfactuals would automatically count as true. For instance the counterfactual "If I were to have green hair then I would have blue hair" is plainly false; but the implied translation "I have green hair $\supset$ I have blue hair" is true simply because the antecedent is false and $0 \supset \ldots = 1$.

In this section we have discussed some of the nuances of English that we intentionally disregard in translation into PL. In real-life communication these nuances can play an important role. Unfortunately, space limitations prevent us from detailing such matters in this text, which emphasizes formal rather than informal logic. For a nice exposition of this topic, including a treatment of "*conversational implication*" (what is tacitly implied by our linguistic conventions), see Ch. 1 of *Understanding Arguments* by R. J. Fogelin, as well as Paul Grice's paper "Logic and Conversation" which is included as an appendix in Fogelin's text. A complete treatment of communication would need to take into account also the various aspects of non-verbal communication and body-language.

EXERCISE 2.6

1. Translate the following PL sentences into English using the dictionary supplied.

H = People will be happy
D = The inflation rate drops
E = People empathize with one another
U = People understand how others feel
S = People are selfish

(a) $U \equiv E$
(b) $(S \vee \sim D) \supset \sim H$
(c) $\sim D \equiv (S \,\&\, \sim U)$
(d) $S \supset \sim (D \vee H)$
*(e) $\sim (H \equiv (U \,\&\, D \,\&\, \sim S))$

2. Given the dictionary of Question 1, symbolize the following into PL.

(a) If people are selfish they will not empathize with one another and will be unhappy.
(b) Unless people are selfish, the inflation rate will drop and they will be happy.
(c) It is false that if the inflation rate drops but people are still selfish, they will be happy.

(d) People are unselfish only if they neither empathize with nor understand one another.

(e) Provided people are unselfish they will not only empathize with one another but will be happy as well.

3. Translate the following into PL using the dictionary supplied.

C = Paul eats the chips
F = Norma has a feed
P = Paul eats the popcorn

(a) Norma has a feed when Paul eats the chips.

(b) It's not the case that Paul eats both the chips and the popcorn.

(c) Paul eats the chips but not the popcorn.

(d) Norma has a feed unless Paul eats the popcorn as well as the chips.

*(e) Given that Paul eats the chips, Paul's eating the popcorn will be a sufficient condition for Norma's going without a feed.

*(f) Unless Norma misses out on her meal, Paul will eat the chips or popcorn but not both.

*(g) For Norma to go hungry it is necessary that Paul eats both the chips and the popcorn.

(h) Norma has a feed only if Paul doesn't eat both the popcorn and the chips.

(i) Paul eats the chips whenever he misses out on the popcorn.

(j) If Paul eats neither the popcorn nor the chips, Norma has a feed.

(k) Norma goes hungry if and only if Paul eats the chips and the popcorn.

(l) Either Paul doesn't eat the chips or he doesn't eat the popcorn or Norma doesn't have a feed.

(m) If Norma has a feed then Paul either goes without the chips or goes without the popcorn.

*(n) Paul's eating the chips is a necessary but not sufficient condition for Norma's missing out on a feed.

4. A = Superman appears
D = Clark Kent disappears
I = Clark Kent is Superman
L = Lois becomes suspicious

Using the above dictionary, translate the following into English.

(a) $(D \& A) \supset L$

(b) $A \supset (\sim L \equiv \sim D)$

Using the above dictionary, symbolize the following.

(c) If Superman appears while Clark Kent doesn't disappear then obviously Kent is not Superman.

*(d) The combination of Superman's appearance with Clark Kent's disappearance is a necessary though insufficient condition for either Lois becoming suspicious or Kent being Superman.

5. Translate the following sentences into PL, providing your own dictionary.

(a) This is an easy one.

(b) This one is harder but not much harder.

(c) Sam sliced the sausage and Sue slid on the slippery slime.

(d) I will be more satisfied if I watch less TV and read more books.

(e) I'm not sure that I like all the examples in this book.

(f) Once you do a bit of logic you find it's both interesting and enjoyable.

*(g) Even though logic is difficult sometimes, you will find it rewarding provided you make an effort.

(h) Neither is Mars a star nor is Alpha Centauri a planet.

(i) Exactly one of Saturn and Neptune lies between Jupiter and Uranus.

*(j) My being a man is a sufficient but not necessary condition for me to be a human.

(k) My brother stayed up watching TV last night till 8, but I beat him 'cause I stayed up till "96".

(l) Every time the six million dollar man lands after jumping from a great height his legs should spear his upper body.

(m) The next program is unimaginative and neither Norma nor Paul nor David nor Linda nor Selena nor I will watch it.

(n) It's false that, I will watch the next program if and only if Norma, Linda and Selena all watch it.

*(o) Given that Paul and David watch the next program but Selena doesn't, it's not true to say that provided Norma doesn't watch it Linda's watching it guarantees that I watch it.

*6. Translate the following into PL using the suggested letters as propositional constants.

(a) To be religious it is neither necessary nor sufficient that you believe in God. (*R*,*B*)

(b) It can't be that just one of Anderson, Belnap and Cantor is a logician. (*A*,*B*,*C*)

(c) In spite of the fact that not only Aquinas but also Descartes produced proofs for God's existence it is clear that, unless he lied about his religious stance, Marx was an atheist. (*A*,*D*,*L*,*M*)

(d) It would be erroneous to assert that only if you are a Buddhist can you be both religious and agnostic; whereas it is true that being irreligious implies that you aren't a Buddhist. (*B*,*R*,*A*)

(e) You work hard at logic and, provided you apply it in everyday life, it will improve both your thinking and your sense of humour. (*W*,*A*,*T*,*H*)

Puzzle 2.

On visiting an old country town Mr C.T. Slicker found a wooden post with a message engraved on it as shown.

Despite his post-doctoral qualifications in linguistics he was unable to decipher the message.

Can you help him?

2.7 SUMMARY

A *wff* of PL can be assembled using only these formation rules: p, q, r, s, t (with or without subscripts) are wffs; if α is a wff so is $\sim \alpha$; if α and β are wffs so is $(\alpha \star \beta)$ where $\star$ is one of $\&$, $\vee$, $\supset$, $\not\equiv$. Later we relaxed (or added to) these rules by allowing propositional constants, different style brackets, and omission of brackets when no ambiguity resulted.

We use p,q,r,s,t as propositional *variables, A, ..., Z* as propositional *constants*, and 1 and 0 as *truth values* (true, false). Our propositional *operators* are as follows.

OPERATOR	**SYNTACTICAL NAME**	**OPERATION NAME**	**USED TO TRANSLATE**
$\sim$	tilde	negation	not
$\&$	ampersand	conjunction	and
$\vee$	wedge	(inclusive) disjunction	or (inclusive)
$\supset$	hook	material implication	if ... then ...
$\equiv$	tribar	material equivalence	if and only if
$\not\equiv$	slashed tribar	exclusive disjunction	... or ... but not both

The operators are defined by the following truth tables:

p	$\sim p$
1	0
0	1

p	q	$p \& q$	$p \vee q$	$p \supset q$	$q \supset p$	$p \equiv q$	$p \not\equiv q$
1	1	1	1	1	1	1	0
1	0	0	1	0	1	0	1
0	1	0	1	1	0	0	1
0	0	0	0	1	1	1	0

Although one column is sufficient for the defining of hook, we have set out two to make it clear that hook is not commutative.

A wff's *main operator* is the last operator inserted in assembling it by the formation rules.

$\&$, $\vee$, $\equiv$, $\not\equiv$ are *commutative* but $\supset$ is not e.g., $p \& q$ is equivalent to $q \& p$ but
$p \supset q$ is not equivalent to $q \supset p$

$\&$, $\vee$, $\equiv$, $\not\equiv$ are *associative* but $\supset$ is not, e.g.,
$p \& (q \& r)$ is equivalent to $(p \& q) \& r$ but
$p \supset (q \supset r)$ is not equivalent to $(p \supset q) \supset r$

A proposition is *atomic* if it contains no other propositions; otherwise it is *compound.* PL-*sentences* contain no propositional variables: they express propositions e.g., $A \& B$. PL-*forms* contain no propositional constants e.g., $p \& q$. In an *explicit* PL-sentence, each propositional constant represents an atomic proposition. In the explicit PL-form of a proposition, each occurrence of the same propositional variable relates to the same atomic proposition. For example, given that A and B are atomic, the explicit PL-form of the proposition $A \& (B \vee A)$ is $p \& (q \vee p)$.

When translating from English to PL we aim for *synonymy* (same meaning) or *equivalence* (same truth conditions), but must sometimes be satisfied with *implied* translations (proposition expressed in English implies that expressed in PL).

Translation into PL may be done in a *top-down* fashion (begin with the main operator of the whole proposition then proceed through the "main operators" of the components, ending with the atomic propositions), or a *bottom-up* fashion (the reverse), or a combination of the two. Brackets are normally required when symbolizing a dyadic operator, or a "not" which has scope over a compound proposition. Clues to bracketing are provided by word links such as those between "either" and "or", "both" and "and", and "if" and "then", as well as by commas.

The translation guide below lists samples of English expressions and the way they are usually translated into PL. In some cases these expressions may need to be translated differently: here your sensitivity to English usage and your purpose for making the translation will be your guides.

$\sim p$	Not p	
	It's not the case that p	
	It's not true that p	
	It's false that p	
	It can't be that p	
$p \& q$	p and q	$p; q$
	Both p and q	$p;$ however q
	p but q	$p;$ nevertheless q
	p although q	$p;$ moreover q
	p even though q	p, yet q
	p in spite of q	p notwithstanding q
	p and also q	p whereas q
	Not only p but also q	p while q
	p as well as q	p because q (but see §2.6)
$p \vee q$	p or q	p and/or q
	Either p or q	At least one of p and q
	p or q or both	Unless p, q
	p unless q	p except when q
$p \supset q$	If p then q	Provided that p, q
	If p, q	On condition that p, q
	p only if q	p is sufficient for q
	p implies q	For q it is sufficient that p
	Whenever p, q	p guarantees that q
	Given that p, q	p only in case that q
	Had p, q	When p, q (but see §2.6)
$q \supset p$	p if q	p is necessary for q
	p if and when q	For q it is necessary that p
	Only if p, q	p provided that q
	p given that q	p in case q
$p \equiv q$	p if and only if q	p is necessary and sufficient for q
	p when and only when q	p just in case q
	If p then q and conversely	

$p \not\equiv q$	p or q but not both
	Exactly one of p and q
	Just one of p and q
$\sim(p \,\&\, q)$	Not both p and q
	p is incompatible with q
$\sim(p \vee q)$	Not either p or q
	Neither p nor q

3

Truth Tables

3.1 INTRODUCTION

Having gone to some pains to develop our new language (PL), and in the process tidied up various matters about communicating in the English language, we now begin to exploit the marvellous clarity and efficiency of PL by using it to simplify the analysis of propositions and propositional relations. In the next chapter we will use it to simplify the analysis of arguments. Once you have learned a few easy rules on how to apply the methods and had some practice at the problems, you will be well on the way to mastering the techniques of Propositional Calculus.

In this chapter we investigate some of the uses to which truth tables may be put. After finding out how to calculate the "main column" of a formula's truth table, we will apply this knowledge to test certain properties of propositions and certain relationships between propositions. As a spin-off, this will enable us to list various "logical truths" which play a key role in later work.

3.2 THE MAIN COLUMN

To simplify later discussion we now introduce the term *"propositional letter"* as a generic term covering both propositional variables and propositional constants. For example, both *"p"* and *"A"* are propositional letters when they are used in PL-wffs. We may thus regard PL-wffs as being composed of three different types of symbols: letters, brackets and operators.

We have already dealt with truth tables for simple formulae with just one operator, but we haven't had much practice with tables for longer formulae. The general procedure for building tables is roughly as follows. We begin by noting the propositional letters in the formula, then writing down the matrix and the formula itself. Columns of truth values for the letters and operators within the formula are then evaluated and placed directly underneath the evaluated symbol. The order of this evaluation is bottom up: it is the same as the order in which the formula would be built up by the formation rules of PL. The final column calculated (known as the *"main column"* because it is under the *main operator*) is then identified by placing an arrow underneath it. Before summarizing this procedure in a formal way, let's look at an example.

Example 1: To compute the truth table for $\sim(p \supset q)$ we begin by writing down the matrix and the expression as shown.

p	q	$\sim(p \supset q)$
1	1	
1	0	
0	1	
0	0	

The wff may be built up from the formation rules as follows:

1.	p	B
2.	q	B
3.	$(p \supset q)$	$1, 2, R \supset$
4.	$\sim(p \supset q)$	$3, R \sim$

We should therefore enter the truth values of the table in the following order: p, q, $(p \supset q)$, $\sim(p \supset q)$. This is done in the table below.

p	q	$\sim(p \supset q)$
1	1	0 1 1 1
1	0	1 1 0 0
0	1	0 0 1 1
0	0	0 0 1 0
		↑

Note that the values of p are placed right underneath p, the values of q below q, the values of $(p \supset q)$ below $\supset$, and the values of $\sim(p \supset q)$ below $\sim$. Since $\sim$ is the main operator, the column under $\sim$ is the *main column* and it gives the values for the expression *taken as a whole*: it is identified by an arrow as shown.

In filling out values under the expression it is *not* necessary to show the values of the letters, because they are already shown in the matrix. Thus the table for $\sim(p \supset q)$ would usually be written:

p	q	$\sim(p \supset q)$
1	1	0 1
1	0	1 0
0	1	0 1
0	0	0 1
		↑

Also, it is *not* necessary to write down an assembly line for the formula every time we construct a truth table. The order of the steps is best worked out mentally by imagining the order in which you would insert the operators in an assembly line for the formula. You may wish to use the following rules of thumb to ensure you evaluate the operators in a correct order.

Rule: Evaluate bracketed expressions before their adjoining operators.

Rule: Evaluate $\sim$ before the other operators unless this breaks the above rule.

Rule: Evaluate consecutive $\sim$ s right to left.

Here are some examples of these rules in action. The numbers underneath the operators indicate the order in which the operators should be evaluated (starting at 1).

$$p \underset{2}{\&} \underset{1}{\sim} q \qquad\qquad \underset{2}{\sim}(p \underset{1}{\vee} q)$$

$$\underset{2}{\sim}(p \underset{1}{\&} q) \underset{3}{\supset} r \qquad\qquad p \underset{3}{\equiv} (p \underset{2}{\vee} (p \underset{1}{\&} q)) \qquad\qquad \underset{3}{\sim} \; \underset{2}{\sim} \; \underset{1}{\sim} p$$

One consequence of the first rule is that inner brackets must be treated before outer brackets (see the fourth example above).

Here now are two harder examples. For each of these there is more than one possible assembly line, and hence more than one correct order for the operators (two samples are given for each). The main operator of a formula is the last operator inserted in any assembly line for the formula. Consequently, the main operator is the final operator evaluated. Since the main operator is unique, the final operator evaluated in the examples below must be the one shown.

$$(p \underset{\substack{2\\4}}{\&} \underset{\substack{1\\3}}{\sim} q) \underset{\substack{5\\5}}{\supset} \underset{\substack{4\\2}}{\sim}(q \underset{\substack{3\\1}}{\vee} p) \qquad\qquad \underset{\substack{3\\5}}{\sim}((p \underset{\substack{1\\1}}{\&} q) \underset{\substack{2\\3}}{\supset} r) \underset{\substack{6\\6}}{\supset} ((p \underset{\substack{5\\4}}{\supset} (q \underset{\substack{4\\2}}{\supset} r)))$$

For any formula that has an operator, its main column will be underneath its main operator. If a formula has no operator (i.e. if the formula is simply a propositional letter) then it has only one column underneath it and this is its main column. In either case, *the main column is the last one calculated.*

The method of constructing truth tables for formulae may now be summarized.

Method: 1. **Write down the matrix and the formula.**
2. **Evaluate the formula in *assembly line order* (see above rules), placing truth columns *directly under* the relevant symbol.**
3. **Identify the main column by an arrow underneath.**

Example 2: Now let's try a harder one: the formula $(p \vee q) \supset (p \,\&\, \sim q)$. We begin in the normal way.

p	q	$(p \vee q) \supset (p \,\&\, \sim q)$
1	1	
1	0	
0	1	
0	0	

Two possible evaluation orders are: $(p \underset{\substack{1\\3}}{\vee} q) \underset{\substack{4\\4}}{\supset} (p \underset{\substack{3\\2}}{\&} \underset{\substack{2\\1}}{\sim} q)$

For this formula, ~ must be evaluated before &, and ⊃ must be evaluated last. This yields the following result.

p	q	$(p$	$\vee$	$q)$	$\supset$	$(p$	$\&$	$\sim$	$q)$
1	1		1		0		0	0	
1	0		1		1		1	1	
0	1		1		0		0	0	
0	0		0		1		0	1	
					↑				

Example 3: Now look at the formula $p \,\&\, q \,\&\, \sim p$.

The absence of brackets is allowable because of the *associativity* of &, i.e.

$(p \mathbin{\&} q) \mathbin{\&} {\sim}p$ (1)

is equivalent to

$p \mathbin{\&} (q \mathbin{\&} {\sim}p)$ (2)

whenever we have a case like this it doesn't matter which of the equivalent forms we choose. Reading the original formula as (1) gives

p	q	$(p \mathbin{\&} q) \mathbin{\&} {\sim}p$
1	1	1 0 0
1	0	0 0 0
0	1	0 0 1
0	0	0 0 1
		↑

while the second reading gives

p	q	$p \mathbin{\&} (q \mathbin{\&} {\sim}p)$
1	1	0 0 0
1	0	0 0 0
0	1	0 1 1
0	0	0 0 1
		↑

Even though the tables are different because of the different orders in which the formula was evaluated, the main columns are in agreement. It is better for the sake of clarity to insert your choice of brackets (as we have done) when drawing a truth table for an associative expression.

Example 4: The expression $p \equiv (q \vee r)$ has three letters. How do we set up the matrix for it?

As mentioned in the previous chapter, the matrix lists all the permutations of the truth values of the letters in the formula. Here the letters are p, q, and r. We know that there are four cases with just p and q; since each of these may be associated with r true and r false there must be eight rows in our matrix. It doesn't really matter in which order we put these rows. However in this book the following order will be adopted:

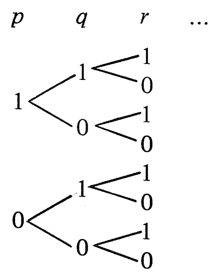
p	q	r
1	1	1
1	1	0
1	0	1
1	0	0
0	1	1
0	1	0
0	0	1
0	0	0

We can now compute the truth table for our formula. Larger truth tables involve no more tricks; they are just longer. Check through the table below to see that you agree with it.

p	q	r	$p \equiv$	$(q \vee r)$
1	1	1	1	1
1	1	0	1	1
1	0	1	1	1
1	0	0	0	0
0	1	1	0	1
0	1	0	0	1
0	0	1	0	1
0	0	0	1	0
			↑	

Matrix Order:

The matrix order adopted in this book is based on a "tree structure" as indicated below. Each pathway from the left to the right of the tree provides one matrix row.



It should be clear that each time a new letter is added the number of rows doubles, since each of the previous rows can be associated with the values 1 and 0 for the new letter.

No. of letters in formula	1	2	3	4	5	...
No. of rows in truth table	2	4	8	16	32	...

This fact may be summarised thus:

A formula with n letters has 2^n rows in its truth table.

For example, a formula with 4 letters will have 2^4 (= 2x2x2x2) or 16 rows in its truth table.

Once we know the number of rows, the matrix for any formula may be systematically filled in from ***right to left*** as follows. Fill the rightmost matrix column with alternating 1's and 0's i.e. 1, 0, 1, 0, ... Now fill the second rightmost column with alternating doubles of 1's and 0's i.e. 1, 1, 0, 0, ... Now fill the third rightmost column with alternating quadruples i.e. 1, 1, 1, 1, 0, 0, 0, 0, ... Continue this "doubling up" procedure for each column added to the left, until all the propositional letters have been catered for.

Check this procedure for yourself on the sample matrices shown below.

p
1
0

p	q
1	1
1	0
0	1
0	0

p	q	r
1	1	1
1	1	0
1	0	1
1	0	0
0	1	1
0	1	0
0	0	1
0	0	0

p	q	r	s
1	1	1	1
1	1	1	0
1	1	0	1
1	1	0	0
1	0	1	1
1	0	1	0
1	0	0	1
1	0	0	0
0	1	1	1
0	1	1	0
0	1	0	1
0	1	0	0
0	0	1	1
0	0	1	0
0	0	0	1
0	0	0	0

Pairing Brackets:

Our rules for evaluating wffs of PL correspond to the following priority convention:

$$\begin{array}{l} p,\ q,\ \ldots \\ (\,) \\ \sim \\ \downarrow\ \&,\ \vee,\ \supset,\ \equiv,\ \not\equiv \end{array}$$

(For simplicity, propositional constants and different style brackets have been omitted.) In mathematics and computing, the dominant practice is to classify dyadic operators into different priority levels e.g., x has higher priority than + so 8 + 4 x 2 = 8 + 8 = 16. Moreover, it is usual to evaluate dyadic operators of equal priority (e.g., +, – or x, ÷) in left-to-right order unless brackets override this e.g., 8 ÷ 4 x 2 = 2 x 2 = 4. In PC however, all dyadic operators have the same priority and no left-to-right order convention is adopted e.g., $p \supset q \vee r$ is ill-formed and cannot be evaluated. As a consequence of this, brackets tend to be more numerous in formulae of PL than in mathematical formulae. With longer PL-wffs this may lead to some difficulty in pairing up brackets or in deciding which bracketed subformula to evaluate next.

A systematic way of pairing brackets is as follows. Begin at the left of the formula and move right until you meet the first right bracket: pair this with the prevous left bracket. Move right until you meet the next right bracket: pair this with the previous unpaired left bracket. And so on. Use this method to check the matching of parentheses in the below example.

$$\sim\underbrace{(\underbrace{(\underbrace{(p \supset \underbrace{(q \vee r)})}\ \&\ s)} \equiv \underbrace{(t \supset q)})}$$

Another way to indicate the order of pairing brackets is to index the brackets as in the below example. Check that the indexing agrees with our pairing order.

$$\overset{5}{(}\,\overset{3}{(}\sim\overset{2}{(}q \supset \sim\overset{1}{(}r\ \&\ \sim p\ \&\ q\overset{1}{)}\overset{2}{)} \vee q\overset{3}{)}\ \&\ \overset{4}{(}\sim p \vee r\overset{4}{)}\overset{5}{)}$$

Once parentheses have been paired, the formula can be made more readable by using brackets of different shape or colour and by deleting outermost parentheses (if any).

This pairing technique can be combined with our evaluation rules to provide the following automatic procedure for evaluating wffs of PL:

1. Substitute truth values for the propositional letters.
2. Move from left to right stopping at the first ")" : the expression between this and the previous "(" is the next subformula to be evaluated. If no brackets were met the whole formula may now be evaluated.
3. Evaluate $\sim$'s.
4. Evaluate dyadic operators.
5. Replace (subformula) or whole formula with its value. If whole formula now evaluated then stop; else go to step 2 (begin move right from the replacement point).

Where different style brackets are used, ")" and "(" in the above procedure should be replaced with "right bracket" and "left bracket" respectively.

NOTES

Some logicians do adopt priority conventions for dyadic operators in propositional logic (e.g., & before $\vee$). Alternative conventions and notations are discussed in detail in Ch. 9.

At step 4 in the procedure above, there will be at most one dyadic operator or multiple occurrences of the same associative dyadic operator. In the latter case any evaluation order may be adopted, though left to-right is usually recommended.

For details of a computer program for evaluating wffs of PL according to the procedure discussed in this section see Halpin, T.A., "PC Formulae Evaluation in BASIC" in *The Australian Logic Teachers Journal* Vol 5 No 2 (1981 Feb).

EXERCISE 3.2

1. For each of the following formulae, place numbers under the operators to indicate the order in which they should be evaluated.
   (a) $\sim(p \supset p)$
   (b) $p \equiv \sim q$
   (c) $(p \supset q) \not\equiv (\sim q \mathbin{\&} p)$
   (d) $p \vee q \vee r$
   (e) $(p \mathbin{\&} q) \vee (r \supset \sim s)$
2. Draw truth tables for the formulae in Question 1.
3. (a) Draw the matrix for a formula containing the variables p, q, r, s, t.
   (b) How many rows are there in the truth table for an expression containing seven propositional letters?
4. For each of the following formulae, index the brackets according to the systematic evaluation order discussed in this section, then evaluate the formula for the following assignments of values to the PVs:

   $p = 0$
   $q = 1$
   $r = 1$
   $s = 0$

   (a) $(((p \mathbin{\&} q) \supset r) \supset p)$
   (b) $\sim((q \vee (p \equiv r)) \supset \sim(r \mathbin{\&} q))$
   (c) $\sim(\sim(\sim q \equiv ((s \vee p) \supset \sim((p \mathbin{\&} \sim q) \vee r))) \supset (p \supset q))$

*5. [For those with some knowledge of computer programming]

(a) Write a computer program to print the matrices for formulae with the following PVs.

(i) p, q

(ii) $p\ q, r\ s\ t$

(Hint: Use nested loops with a step of -1)

(b) If your computer language has logical operators for AND, OR, NOT write a program to print the truth table for the following formula:

$$(p \vee q) \vee \sim(q \mathbin{\&} r)$$

3.3 CLASSIFYING PL-FORMS

Once the main column of a PL-form has been calculated, the formula may be classified into one of three types according to whether just 1's, just 0's, or both 1's and 0's are present.

PL-form	**Main-column values**
tautology	**all 1**
contradiction	**all 0**
contingency	**some 1, some 0**

Example 1: Compute the main-column of the following PL-forms, and then classify them on that basis.

$$p \supset p, \quad p \mathbin{\&} \sim p, \quad p \mathbin{\&} p$$

p	$p \supset p$	$p \mathbin{\&} \sim p$	$p \mathbin{\&} p$
1	1	0 0	1
0	1	0 1	0
	↑	↑	↑

∴ $p \supset p$ is tautologous
$p \mathbin{\&} \sim p$ is self-contradictory
$p \mathbin{\&} p$ is contingent

Notice how a *common matrix* was used for the three formulae, to save space. Note also the adjectival versions of the three classifications.

It is not always necessary to complete the main column when testing a PL-form. As soon as we get one 0 we know it is not a tautology. As soon as we get one 1 we know it is not a contradiction. As soon as we get one 1 and one 0 we know it is a contingency.

Example 2:

p	q	$p \supset \sim q$
1	1	0 0
1	0	1 1
0	1	
0	0	
		↑

∴ $p \supset \sim q$ is contingent.

Once you've done a couple of truth tables you'll find them quite easy (provided you follow the rules!) With longer formulae you might like to save some work by using a few *short cuts*, as discussed below.

When evaluating a formula or sub-formula whose main operator is &, ∨ or ⊃ we will often be able to assign a value as soon as know the value of *one* of the operands. Consider for instance the following conjunction:

$$\begin{array}{l} p \;\&\; [q \supset (r \;\&\; s)] \\ 0 \end{array}$$

Suppose, as is indicated above, that we know that $p = 0$ on the row being evaluated. Recall that a conjunction = 1 iff both its conjuncts = 1. The fact that $p = 0$ then, implies that the conjunction will be false regardless of what value $[q \supset (r \;\&\; s)]$ might have. So we may write 0 under the & without having to evaluate the right conjunct.

$$\begin{array}{l} p \;\&\; [q \supset (r \;\&\; s)] \\ 0\;0 \end{array}$$

This example is an instance of the general result that: if the left conjunct is false the whole conjunction is false. Let us abbreviate this as: 0 & ... = 0.

We call the above result a *"one operand evaluation rule"* since it allows a dyadic expression to be evaluated from the value of one operand. There are six such rules for our operators, as summarized below. Their justification is given on the right. Do you see why there are no such rules for ≡ and ≢ ?

$$\left.\begin{array}{l} 0 \;\&\; \ldots = 0 \\ \ldots \;\&\; 0 = 0 \end{array}\right\} \text{A conjunction} = 0 \text{ if at least one conjunct} = 0$$

$$\left.\begin{array}{l} 1 \vee \ldots = 1 \\ \ldots \vee 1 = 1 \end{array}\right\} \text{A disjunction} = 1 \text{ if at least one disjunct} = 1$$

$$\left.\begin{array}{l} 0 \supset \ldots = 1 \\ \ldots \supset 1 = 1 \end{array}\right\} \alpha \supset \beta = 1 \text{ if either } \alpha = 0 \text{ or } \beta = 1$$

Notice that such rules apply only for certain values of the operands. For example, if one conjunct is evaluated as 1 we have to determine the value of the other conjunct as well before we can evaluate the whole conjunction.

Example 3: Classify $(p \;\&\; {\sim} q) \supset (p \vee {\sim} r)$ as a tautology, contradiction or contingency.

p	q	r	$(p$	$\&$	${\sim}q)$	$\supset$	$(p$	$\vee$	${\sim}r)$
1	1	1				1		1	
1	1	0				1		1	
1	0	1				1		1	
1	0	0				1		1	
0	1	1		0		1			
0	1	0		0		1			
0	0	1		0		1			
0	0	0		0		1			
						↑			

∴ tautology

> On the top 4 rows $p = 1$: therefore $(p \vee {\sim} r) = 1$. On the bottom 4 rows $p = 0$: hence $(p \;\&\; {\sim} q) = 0$. The main column follows immediately by applying the rules $\ldots \supset 1 = 1$, and $0 \supset \ldots = 1$.

In later sections you will find the terms "tautology", "contradiction" and "contingency" used to classify propositions. Though there is a connection with the use of these terms to classify PL-forms, the connection is not a simple one. The differences will be explained later.

NOTES

The stipulative definition given in logic to the word "tautology" is significantly different from the dictionary definition "saying the same thing twice over in different words".

In this section we have dealt with the classification of PL-forms, but not the classification of propositions. The former problem may be regarded as answerable within 2-valued formal semantics (in formal semantics 1 and 0 are just uninterpreted values rather than "true" and "false"). The latter problem raises other questions (e.g., the adequacy of PL to display the relevant structure of propositions) and should be regarded as a distinct problem.

As we use the terms, the *language* PL consists of the infinitely many wffs which may be assembled by the formation rules of PL; the *system* PC includes the formal semantics whereby wffs of PL may be assigned values and divided into tautologies and non-tautologies.

For reasons to be discussed later, some authors use the term "indeterminate" rather than "contingent" when classifying forms. After considerable deliberation however, we have decided to retain the term "contingent" for this use, partly because it seems firmly entrenched in the literature, but mostly because from the point of view of formal semantics it is a fully determinate matter whether the value of a PL-form is contingent (dependent) on the values given to its PVs. The distinction between "contingent" as applied to forms and "contingent" as applied to propositions will be carefully drawn later.

EXERCISE 3.3

1. Classify each of the PL-forms in Question 1 of Exercise 3.2 as a tautology, contradiction or contingency.

2. Using either complete or shortened truth tables, classify each of the following as tautologous, self-contradictory or contingent. To save writing, a common matrix may be used to test several formulae with the same PV's.

   (a) $\sim p$
   (b) $\sim\sim p$
   (c) $p \vee p$
   (d) $p \equiv p$
   (e) $p \not\equiv p$
   (f) $\sim(p \vee p)$
   (g) $\sim(p \,\&\, p)$
   (h) $p \supset \sim p$
   (i) $p \equiv \sim p$
   (j) $p \,\&\, q$
   (k) $q \supset p$
   (l) $\sim p \equiv q$
   (m) $p \supset (q \supset p)$
   (n) $\sim p \supset (p \supset q)$
   (o) $p \supset (p \vee q)$
   (p) $(p \supset q) \,\&\, (p \,\&\, \sim q)$
   (q) $p \equiv (q \not\equiv r)$
   (r) $\sim[(p \,\&\, q) \supset (p \vee r)]$
   (s) $\sim\sim q \equiv \sim[r \supset (p \supset \sim q)]$
   *(t) $\sim(r \vee p) \supset (q \vee \sim[(p \not\equiv q) \,\&\, \sim p])$

*3. By inspection of the formula, and logical deduction, classify the following formula without the aid of a truth table.

$$\sim(p \,\&\, \sim(q \vee p))$$

Outline the steps in your deduction.

*4. If you managed Question 5 (b) of Exercise 3.2, modify your computer program to include a test as to whether the formula tested there is a tautology, and have the result printed out.

3.4 MODAL PROPERTIES OF PROPOSITIONS

In this section the logician's notion of *"possible worlds"* is briefly explained with the aid of examples. This concept will then be used to define a classification scheme for *propositions* (as distinct from PL-forms) in terms of their *"modal properties"*. The terms "contradiction" and "contingency" will appear again, though with a somewhat different connotation as they apply to propositions. Finally we will use our educated intuitions on what counts as possible, to determine the modal properties of a variety of propositions expressed in English. In the next section, we will investigate how truth tables may be used to assist us in such determinations.

Let's begin now with the notion of possible worlds. Clearly, our universe, the *actual world*, is one possible world. Besides the actual universe however, there are infinitely many worlds which *might* have been. We can imagine many such possible worlds: a world with no Earth; a world where Earth has three moons; a world where Mars is inhabited by little green men; etc. You can have fun inventing some of your own possible worlds. And no doubt there are many possible worlds beyond the reach of our limited imagination.

Not every world is possible however. For instance a world in which Earth both does and doesn't exist is an *impossible world.*

Now look closely at the following propositions. Each of these is true, but in a very special way.

If it is raining then it is raining. (1)
All animals are animals. (2)
One plus one equals two. (3)
All bachelors are male. (4)

None of these can be false under any circumstances. They *must* be true i.e. they are true in all possible worlds. We shall call such propositions *necessary truths* or *logical truths*.

Now what do you notice about the following propositions?

It's raining and it's not raining. (5)
Not all animals are animals. (6)
One plus one equals three. (7)

None of these can be true under any circumstances. They *must* be false i.e. they are false in all possible worlds. Propositions like these describe logically impossible states of affairs, and are known as *necessary falsehoods* or *contradictions.*

Note that we are using the terms "possible" and "impossible" in a *logical* rather than a *physical* sense. An example should clarify this.

The space ship travelled at twice the speed of light. (8)

While the laws of physics may imply that (8) can't happen in the actual world, we can quite consistently imagine possible worlds where the laws are different and in which (8) is true. So even if physically impossible, (8) remains logically possible: it is not a contradiction.

Like (8), most propositions we meet in everyday talk are neither necessarily true nor necessarily false. Here are some examples:

It's raining. (9)
World War II ended in 1945. (10)
There will be no World War III. (11)

Each of these is true in some possible worlds and false in others i.e. the truth value of each is contingent (dependent) on the possible world being discussed. Such possibly true and possibly false propositions will be referred to as *contingent propositions* or *contingencies.*

Both necessary truths and contradictions are *non-contingent* propositions. Some books use "necessary" as a synonym for "non-contingent"; in this text however the word "necessary" will be taken to mean "necessarily true". Thus, contradictions will not be counted as necessary.

Any proposition must fall into one, and only one, of the three categories shown below.

Propositions
- **Necessary Truth: true in all possible worlds**
- **Contradiction: false in all possible worlds**
- **Contingency: true in some possible worlds and false in others**

In mediaeval logic, terms like "necessity", "contingency" etc. were used to indicate the *mode* (or manner) in which a proposition could be true or false. From our viewpoint, describing propositions as being necessary truths, contradictions or contingencies indicates the mode in which their truth values are distributed across the set of all possible worlds: such properties will be called *modal properties* to distinguish them from other propositional properties (such as being true, false, negations, conjunctions etc.)

It needs to be emphasized that when we say there "are" many possible worlds we do not mean they exist in the same way as the actual world. The actual world includes the totality of all events past, present and future (including anything going on at the other end of black holes!). Each possible world represents one way which the actual world might have been. Thus all possible worlds except the actual one exist only as abstract entities (in a similar way to numbers, sets and propositions): we use them primarily as an aid to unifying and understanding logical concepts.

NOTES

Although we have taken the term "possible world" to be understood through paradigm examples, we need at least to agree that in any possible world propositions must have exactly one of two truth values. In addition, we will take it for granted that each possible world contains at least one item. Moreover, when we speak of the *same* proposition taking on truth values in different possible worlds we are assuming that adequate "transworld identity" conditions could be specified to make sense of all this. Consider for instance the contingently true proposition expressed by "Pat Suppes smokes cigars". We need a way of fixing the referent "Pat Suppes" across possible worlds without his cigar smoking being an identifying characteristic. Logicians have proposed various theories (such as "sense" and "causal" theories) to perform this task. In addition to this we need a way of dealing with explicit reference to the actual world. Consider for instance the sentence "In the actual world, Pat Suppes smokes cigars". Under normal construals, this would be taken to say no more than the earlier sentence "Pat Suppes smokes cigars '. So the proposition expressed is false in some possible worlds. This leads to the apparent contradiction that "in some possible world it is false that in the actual world Pat Suppes smokes cigars". To get ourselves out of a mess here it is sufficient to stipulate that reference to "the actual world" is taken to indicate reference to the possible world in which the proposition's truth value is being evaluated i.e. "for" a possible world, "the actual world" *is* that possible world.

EXERCISE 3.4

1. Classify each of the following propositions as necessarily true, self-contradictory, or contingent.
   (a) Australia has a population of over 13 million.
   (b) $3 + 7 = 11$
   (c) Bachelors are married.
   (d) Bachelors are unmarried.
   (e) It's sunny.
   (f) It's sunny or it's not sunny.
   (g) It's not the case that it's either sunny or not sunny.
   (h) A square has four sides.
   (i) If an apple is green then it has a colour.
   (j) Every thing is identical to itself.
   (k) Politicians are honest.
   (l) Numbers have colour.
   (m) Material objects have location in space.
   (n) If Alan is taller than Bill then Bill is shorter than Alan.
   (o) Alan is taller than Bill.
   (p) It is necessarily true that if it's sunny then it's sunny.
   (q) If Alan knows that $1 + 1 = 2$ then Alan believes that $1 + 1 = 2$.

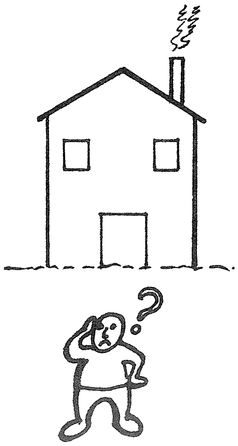
2. Classify each of the following general types of propositions as a necessary truth, contradiction, or contingency.
   (a) the negation of a necessary truth
   (b) the negation of a contradiction
   (c) the negation of a contingency
   (d) the conjunction of a necessary truth and a contradiction
   (e) the disjunction of a necessary truth and a contradiction

3. For each of the following state whether or not it is true.
   (a) Everything which is physically possible is logically possible.
   (b) Everything which is logically possible is physically possible.
   (c) Everything which is physically impossible is logically impossible.
   (d) Everything which is logically impossible is physically impossible.
   (e) In this book, "possible" will be used without qualification to denote logical possibility.

4. In everyday speech, modal concepts are often expressed by "may" and "can". Both these words are sometimes used to convey physical possibility and sometimes mere logical possibility. In addition, "may" is sometimes used to indicate that something is "allowed"; strictly speaking, "can" should not be used in this sense, but by common abuse of the English language it often is. We have to be on our toes to interpret such modal words correctly in various contexts. For each of the following indicate the sense(s) in which these words are being used.

(a) It may rain.
(b) You can pass the test if you try.
(c) People can not jump over tall buildings.
(d) I may win the lottery.
(e) 1 + 1 can not equal 3.
(f) You can talk during the exam but you may not.
*(g) Tom may eat the last piece of cake.

Puzzle 3



Dr. What is puzzled.
He left the front door of his house, walked 10 km due south then 10 km due east, then 10 km due north and found himself back at his front door! Given that his front door is not 10 km wide, does Dr. What live in a possible world?
Explain.

3.5 ASSESSING MODAL PROPERTIES BY TRUTH TABLES

In this section we deal with truth tables for propositions rather than forms, and use them to help determine the modal properties of propositions. We begin by making deliberate use of the concept of possible worlds to specify the way in which truth tables for propositions are to be interpreted: this viewpoint is fundamental to much of the later work and should be well understood.

Given that we have an accurate PL-symbolization for the proposition (and this will usually be assumed) we will take it that *each row of a proposition's truth table refers to just that set of possible worlds in which the matrix assignments are satisfied* (allowing that in certain cases the set may be empty). Since the matrix includes all the permutations of truth values, *the rows collectively cover the set of all possible worlds.*

Since necessary truths are true in all possible worlds, and contradictions are false in all possible worlds, truth tables immediately provide a means of detecting certain types of necessary truths and contradictions. Consider the following proposition for example.

Either reincarnation occurs or it doesn't. (1)

Using the dictionary

R = Reincarnation occurs

we may symbolize (1) as

$R \vee {\sim}R$ (1a)

Since R is atomic, "$R \vee \sim R$" is the explicit sentence for (1) in PL. Let us agree to define the *explicit truth table* of a proposition as the truth table in which it is represented by its explicit sentence (in PL). Thus the explicit truth table for (1) is:

R	R	$\vee$	$\sim R$
1		1	0
0		1	1
		↑	

The first row of values considers the possible worlds in which R is true i.e. the set of possible worlds in which reincarnation does occur: $R \vee \sim R$ is true here. The second row considers the possible worlds in which R is false: $R \vee \sim R$ is true here also. So $R \vee \sim R$ is true in all possible worlds: it is a necessary truth.

Clearly, if the values in the main column of a proposition's explicit truth table are all true, that proposition will be a necessary truth.

Now consider the following proposition.

Alice and Bernie are both happy or they're not both happy. (2)

Using the atomic dictionary

A = Alice is happy
B = Bernie is happy

we may symbolize (2) as

$$(A \,\&\, B) \vee \sim(A \,\&\, B) \tag{2a}$$

So the explicit truth table for (2) is:

A	B	$(A \,\&\, B)$	$\vee$	$\sim$	$(A \,\&\, B)$
1	1	1	1	0	1
1	0	0	1	1	0
0	1	0	1	1	0
0	0	0	1	1	0
			↑		

A glance at the main column reveals that (2) is true in all possible worlds: so it is a necessary truth. Could we have arrived at the same result by using a non-explicit truth table? Let's try the following dictionary:

H = Alice and Bernie are both happy

(2) may now be represented as

$$H \vee \sim H \tag{2b}$$

and the following truth table constructed for it.

H	H	$\vee$	$\sim H$
1		1	0
0		1	1
		↑	

Row 1 of this table refers to just those possible worlds where H is true i.e. where A and B are both true. So row 1 of this table relates to the same set of worlds as row 1 on the explicit truth table does. Row 2 of this table refers to just those possible worlds where H is false i.e. where $(A \,\&\, B)$ is false. So row 2 of this table relates to the same set of worlds as those collectively represented by rows 2, 3 and 4 of the explicit truth table. In other words the total set of worlds represented by the two rows of this table is precisely the same as the total set of worlds represented by the four rows of the explicit truth table. But this should not come as a surprise. Any proposition, atomic or otherwise, must be either true or false, and both these alternatives are considered for the matrix propositions in any truth table. So the rows of any truth table, explicit or otherwise, will collectively cover the set of all possible worlds. So in the case of proposition (2), finding the main-column of its "$H \vee \sim H$" table to be universally true will suffice to establish the necessary truth of (2).

We may now state the following general result. *If a proposition has **any** truth table where its main-column values are all 1, then that proposition is a necessary truth.*

It will also be clear from the truth tables above that the sentences "$R \vee \sim R$", "$(A \& B) \vee \sim(A \& B)$" and "$H \vee \sim H$" are instances of tautologies i.e. they have tautologous forms. Let us agree that *if a sentence expressing a proposition has a particular form then the proposition has that form too.* With this understanding, we may state that if a proposition has a tautologous form then that proposition is a necessary truth. It will simplify things quite a bit if we now extend the application of the term "tautology" to sentences and propositions as well as forms. Let us agree that *a PL-sentence or a proposition is a tautology iff it has a tautologous form.* For example, the form "$p \vee \sim p$", the sentence "$R \vee \sim R$" and the proposition $R \vee \sim R$ are all tautologies. The general result in the previous paragraph may now be restated as follows: *if a proposition is a tautology then it is a necessary truth.*

Now let us consider how truth tables may help to detect contradictions. Remember that a proposition is a contradiction iff it is false in all possible worlds. Since the rows of any truth table collectively refer to precisely the set of all possible worlds, the following result holds: *if a proposition has **any** truth table where its main-column values are all 0, then that proposition is a contradiction.*

Take for example the following proposition:

Alice is happy and Alice is not happy. (3)

Its explicit truth table is:

A	A	&	$\sim A$
1		0	0
0		0	1
		↑	

As the main column shows, (3) is false in all possible worlds, and hence is self-contradictory.

Now consider the following contradiction:

At least one of Alice and Bernie is happy, but neither is happy. (4)

We may show (4) is self-contradictory by producing its explicit truth table:

A	B	$(A \vee B)$	&	$\sim$	$(A \vee B)$
1	1	1	0	0	1
1	0	1	0	0	1
0	1	1	0	0	1
0	0	0	0	1	0
			↑		

As with (2) however, a less than explicit truth table will sometimes do the trick. Choosing the dictionary

L = At least one of Alice and Bernie is happy

the following truth table for (4) may be constructed to prove that it is a contradiction.

L	L	$\&$	$\sim L$
1		0	0
0		0	1
		↑	

From our earlier work on classifying PL-forms it is clear that (3) and (4) have self-contradictory forms e.g., (3) has the form $p \ \& \sim p$. In general, *if a proposition has a self-contradictory form then that proposition is a contradiction.*

Consider now the proposition

Either Sue is not at school or James is at home. (5)

symbolized in PL as $\sim S \vee J$. Its explicit truth-table is:

S	J	$\sim$	S	$\vee$	J
1	1	0		1	
1	0	0		0	
0	1	1		1	
0	0	1		1	
				↑	

Since we know as a matter of fact that S and J can be true or false independently of each other (and hence each row of the matrix is possible), the presence of both 1's and 0's in the main column indicates that (5) is true in some possible worlds and false in some others. So (5) is a contingent proposition. Moreover, (5) has, as its explicit PL-form, the contingent form $\sim p \vee q$. In general, *if a proposition is contingent then its explicit PL-form is contingent.* The converse result however does not hold. Having an explicit PL-form that is contingent does *not* guarantee that the proposition itself is contingent. For example, consider the necessary truth.

All animals are animals. (6)

This is an atomic proposition, so the best we can do in PL is to represent it by the sentence "A". So the explicit (and only) truth table for (6) is:

A	A
1	1
0	0

The main-column values are not all 1, so the fact that (6) is a necessary truth will not be revealed by a truth table. This sort of thing can happen with compound propositions even when the atomic propositions involved are contingent. Here is an example of this:

If John has eight children then he has more than six children. (7)

Using the dictionary

E = John has eight children
S = John has more than six children

we obtain the following explicit truth table:

E	S	$E \supset S$
1	1	1
1	0	0
0	1	1
0	0	1

Again we have a necessary truth whose main-column values are not always 1. So some necessary truths are not tautologies.

As with necessary truths, not all contradictions can be detected by truth tables. For instance, propositions (8) and (9) are contradictions yet the main-columns of their explicit truth tables contain at least one 1.

Not all animals are animals. (8)

A	$\sim A$
1	0
0	1

John has eight, but no more than six, children. (9)

E	S	$E \ \& \sim S$
1	1	0 0
1	0	1 1
0	1	0 0
0	0	0 1
		↑

Cases like these arise because sometimes a truth table row may refer to no possible world at all. This can happen when the atomic propositions are either non-contingent (e.g., (6), (8)) or non-contingently related (e.g., (7), (9)). We will have more to say about this in the section on "possible-truth tables", but for now the important point to grasp is that if the main column of a proposition's truth table contains a mixture of 1's and 0's this does *not* imply that the proposition is contingent: it could be contingent (e.g., (5)) but it could instead be a necessary truth (e.g., (6), (7)) or a contradiction (e.g., (8), (9)).

At this point it will be appropriate to introduce some terminology to formalize our discoveries and indicate how propositions may be classified by means of truth tables.

Some propositions are necessarily true because of their structure within PL i.e. they are *true for all assignments of truth values to their atomic components.* We have called this type of proposition a *tautology*. Since a tautologous proposition can be demonstrated to be a necessary truth by PC methods, it is also appropriately called a *PC-necessary truth* or a *PC-necessity.* A proposition is a tautology iff the main-column values of its explicit truth table are all 1. To prove a proposition is a tautology however, it is sufficient to find *a* truth table for it where the main-column values are all 1 (remember example (2b)). Some necessary truths, like (1) and (2), are tautologies, and others, like (6) and (7), are not.

Some but not all contradictions will be *false for all assignments of truth values to their atomic components:* these are the *PC-contradictions* and the main-column values of their explicit truth tables will all be false. To prove a proposition is a PC-contradiction, it is sufficient to find *a* truth table for it where the main-column values are all 0. Some contradictions, like (3) and (4), are PC-contradictions and others, like (8) and (9), are not.

What about a proposition that is *true for some assignments of truth values to its atomic components and false for other assignments?* The main-column values of its explicit truth table will be a mixture of 1's and 0's. As we have seen, it could be a contingency, or a necessary truth, or a contradiction. In order to find out which, we need to "look inside" the atomic propositions involved. Since the internal structure of atomic propositions cannot be displayed within PL, the precise modal status of such a proposition cannot be determined within PC. It will be appropriate therefore to describe such a proposition as being *PC-indeterminate*.

Our main results for classifying propositions by means of truth tables may now be summarized as below.

Propositions
- **tautology**
- **PC-contradiction**
- **PC-indeterminacy**

A proposition is a *tautology* iff it has *a* truth table for which the main-column values are all 1

A proposition is a *PC-contradiction* iff it has *a* truth table for which the main-column values are all 0

A proposition is *PC-indeterminate* iff the main column of its *explicit* truth table has some 1's and some 0's.

This classification of propositions may be related to our earlier classification of PL-forms as follows. A proposition is a tautology iff it has *a* tautologous PL-form. A proposition is a PC-contradiction iff it has *a* self-contradictory PL-form. A proposition is PC-indeterminate iff its *explicit* PL-form is contingent: such a proposition may be a contingency, necessary truth or contradiction.

When dealing with explicit truth tables it is not always necessary to complete the main column to prove that a proposition is PC-indeterminate. As soon as we get one 0 we know it is not a tautology. As soon as we get one 1 we know it is not a PC-contradiction. So as soon as we get at least one 1 and at least one 0 we know it is PC-indeterminate. This short cut is used in the example below.

Example: What do truth tables indicate about the modal status of the following proposition?

If there is a Third World War the human race will not survive.

Dictionary: W = There is a Third World War
S = The human race survives

Explicit truth table:

W	S	$W \supset \sim S$
1	1	0 0
1	0	1 1
0	1	
0	0	
		↑

∴ the proposition is PC-indeterminate.

NOTES

The notion of possible worlds has played a fundamental role in our treatment of modal properties and truth tables. This approach was motivated by Bradley and Swartz's logical milestone, *Possible Worlds*, which should be consulted by anyone interested in exploring the philosophical basis of logic.

The class of PC-indeterminate propositions is still referred to by many authors simply as "contingent". We have defined a proposition to be contingent iff it is true in just some possible worlds; if this definition is accepted then one must also accept the existence of PC-indeterminate propositions which are non-contingent.

With some misgivings we have allowed "tautology" to have reference across forms, sentences and propositions. Such wide application seems firmly entrenched in the literature, and it does simplify discussion of general procedures. Tautologous propositions may be distinguished from tautologous sentences and forms by the term "PC-necessity". We have deliberately avoided the terms "PC-valid" or "PC-true" as these often blur for students the important distinctions between propositions and arguments or between truth in one model and truth in all models.

EXERCISE 3.5

1. Classify each of the following propositions as a tautology, PC-contradiction or PC-indeterminacy. You may interpret the sentences straightforwardly and in terms of our truth functional operators. First try to determine the answer mentally using your logical intuitions, then check your answer with the truth table test. The following dictionary is suggested:

C = John has a car.
R = John has a red car.
B = John has a blue car.

(a) John has a car.
(b) Either John has a car or he doesn't.
(c) John has a car and he doesn't have a car.
(d) If John has a car then he has a car.
(e) If John has a car then he has a red car.
(f) If John has a red car then he has a car.
(g) John has a car if and only if he has a car.
(h) John has neither a red car nor a blue car.
(i) If John has a red car and a blue car then he has a red car.
(j) John has a red car only if he has either a red or a blue car.
*(k) If John has a car then he doesn't have a car.
(l) John has a car, and if he has a car then he doesn't have a car.
(m) John has a car if and only if it's false that John doesn't have a car.
*(n) If John has a red car then either he has a blue car or he doesn't have a blue car.
*(o) If John has a car and doesn't have a car, then he has a red car.

2. (a) Determine the modal status of the following proposition with the most efficient dictionary you can think of.

John has a red car and a blue car if and only if he doesn't have both
a red car and a blue car.

(b) Which, if any, of the following PL-indeterminate propositions is non-contingent? If you find one, classify it as necessarily true or self-contradictory.

(i) John has a car but he doesn't have a blue car.
(ii) John has a blue car but he doesn't have a car.

(c) One of the PC-indeterminacies in Question 1 is non-contingent. Find it and classify it as a necessary truth or a contradiction.

*3. (a) Try to determine mentally what the modal status of the following piece of legalistic jargon is. Then choose a suitable dictionary (4 propositional constants will suffice) and use truth tables to check your answer.

> This policy will cover you for damage, unless the damage is directly or indirectly, proximately or remotely, occasioned by or contributed to by either war, insurrection or convulsion of nature, or by situations involving not only no convulsion of nature but also the absence of war and insurrection.

(b) For what situations does this policy provide cover?

3.6 POSSIBLE-TRUTH TABLES

In the previous section we saw that, when confronted with a PC-indeterminate proposition, standard truth tables provided no means of deciding in which of the three general modal classes (necessary truth, contradiction, or contingency) it belonged. In this section "*possible-truth tables*" are introduced to provide a more complete method of assessing modal properties. Later they will be used to determine modal relations (including validity of arguments) and to facilitate the solution of logical puzzles.

As we know, each row of a standard truth table for a proposition refers to just that set of possible worlds which satisfy the matrix assignments. Sometimes the set may be empty i.e. the matrix assignments are impossible. This happens when either a single assignment is impossible or the combination of assignments is impossible. We met some cases like this in the previous section. What makes possible-truth tables more special than standard truth tables is that they never have impossible assignments. *In possible-truth tables each row is possible* i.e. each matrix permutation is "possibly-true": it is satisfied in at least one possible world.

We obtain possible-truth tables by writing down a standard truth table for the proposition and crossing off the rows which are impossible. We will indicate such a crossing off by placing a "x" to the left of the matrix row. In order to cross off such a row we need to "look inside" the matrix propositions and use either a more comprehensive logic (e.g., the Quantification Theory to be discussed in Part Two) or our own intuitions (these will usually suffice) to answer the following question: is there a possible world which satisfies these assignments? In doing this we go outside the bounds of the formal propositional calculus.

Let's try this out now on some PC-indeterminacies from §3.5. First consider the proposition

If John has eight children then he has more than six children. (1)

Using the dictionary

E = John has eight children.
S = John has more than six children

we obtain the following possible-truth table

	E	S	$E \supset S$
	1	1	1
×	1	0	0
	0	1	1
	0	0	1
			↑

i.e.

E	S	$E \supset S$
1	1	1
0	1	1
0	0	1
		↑

We reject row 2 because there is no possible world where John has eight children but does not have more than six children.

With a standard truth table, a row is present *if* it represents some possible world. With a possible-truth table, a row is present *if and only if* it represents some possible world. In this way *the set of rows in a possible-truth table indicates* ***exactly*** *the set of all possible worlds*. Consequently the following necessary and sufficient condition can be laid down for necessary truths. *A proposition is a necessary truth iff it has a possible-truth table where its main-column values are all 1.* For example, the possible-truth table above shows that proposition (1) is a necessary truth.

Here's an even easier one:

Not all animals are animals. (2)

Using the dictionary

A = All animals are animals

we get the following possible-truth table

	A	$\sim A$
	1	0
×	0	1

i.e.

A	$\sim A$
1	0

We reject row 2 because there is no possible world where not all animals are animals. Because of the exact matching between the rows of a possible-truth table and the set of all possible worlds, *a proposition is a contradiction iff it has a possible-truth table where its main-column values are all 0.* So the above table allows us to classify (2) as a contradiction. This case was so trivial that the logical intuition we used to reject row 2 could have been used to classify (2) as a contradiction without even bothering to produce any sort of truth table. In more complicated cases however, the possible-truth table method will make life easier.

As a final example, take the proposition

Either Sue is not at school or James is at home. (3)

Using the dictionary

S = Sue is at school
J = James is at home

we obtain the following possible-truth table

S	J	$\sim S$	$\vee J$
1	1	0	1
1	0	0	0
0	1	1	1
0	0	1	1
			↑

Since each of the rows is possible, the possible-truth table is the same as the standard truth-table. Because possible-truth table rows match exactly the set of all possible worlds, *a proposition is a contingency iff it has a possible-truth table where the main column contains a mixture of 1's and 0's.* Thus the above possible-truth table allows us to classify (3) as contingent.

Once a possible-truth table has been constructed then, the modal status of the proposition may be determined as follows.

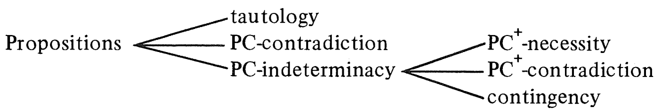
Proposition	Main-column values of possible-truth table
necessary truth	all 1
contradiction	all 0
contingency	some 1, some 0

NOTES

Our possible-truth tables are based on the "corrected truth tables" developed by Bradley and Swartz in their ***Possible Worlds.*** Note that while possible-truth tables may be constructed for propositions, they cannot be constructed for forms.

In practice, besides crossing off impossible rows with "x" it may be helpful to rule a line through them in pencil.

Propositions which are necessarily true or necessarily false but which need a more powerful system than PC to demonstrate this may be called PC^+-necessities and PC^+-contradictions respectively. These, together with contingencies, form the class of PC-indeterminacies.



Provided our logical prowess enables their construction, possible-truth tables go beyond standard truth tables in allowing the type of PC-indeterminacy to be specified, and in so doing allow any proposition to be specified as a necessary truth, contradiction or contingency.

In question 1 of the exercise below, and on some later occasions, we use capital letters to denote propositions which obey a particular restriction but which are otherwise unidentified (the details of the dictionary are withheld). Such letters may be regarded either as "liberated propositional constants" or "restricted propositional variables" when used like this. A standard propositional constant may be viewed as the limiting case of a propositional variable whose range of propositions has been reduced to a single proposition.

EXERCISE 3.6

1. Given three particular propositions A, B and C for which every permutation of truth values is possible, use tables to assess the modal status of the following propositions.
   (a) $A \vee {\sim}(A \,\&\, A)$
   (b) ${\sim}A \supset (B \supset A)$
   (c) $A \,\&\, {\sim}(B \equiv {\sim}C)$
   (d) ${\sim}(A \supset B) \,\&\, {\sim}(C \vee A)$
   (e) $(C \not\equiv {\sim}A) \supset {\sim}(A \equiv B)$

2. Use possible-truth tables and the dictionary provided to assess the modal status of the following propositions.

   C = Pat smokes cigars
   P = Pat smokes a pipe
   S = Pat smokes

   (a) If Pat smokes cigars then Pat smokes.
   (b) If Pat smokes then Pat smokes cigars.
   (c) Either Pat smokes cigars or he doesn't smoke.
   (d) Although Pat smokes cigars, he doesn't smoke.
   (e) Pat smokes if he smokes either cigars or a pipe.
   (f) Unless Pat smokes a pipe, Pat smokes if and only if he smokes cigars.

(g) Either Pat smokes, or he smokes a cigar, but he doesn't both smoke and smoke a cigar.
(h) In spite of Pat's smoking cigars, he doesn't smoke a pipe, and it is the case that he smokes only if he smokes a pipe.
*(i) Only if Pat smokes neither cigars nor a pipe will it be true that if Pat smokes he doesn't smoke a cigar.

3.7 MODAL RELATIONS

In this section we study the nine most important modal relations that can exist between two propositions: equivalence; contradictoriness; implication; converse implication; inconsistency; contrariety; subcontrariety; indifference; and consistency. The first five of these relations may often be detected with standard truth tables; for the latter four relations, possible-truth tables are usually required.

Most of these relations have been introduced earlier in a fairly informal manner. Here we use the possible worlds framework to define them exactly and indicate how they may be tested with the aid of tables.

Given any propositions p and q, we define necessary equivalence in the following way.

p is *necessarily equivalent* to q iff p has the same truth value as q in all possible worlds

Necessary equivalence, also known as "logical equivalence", is much stronger than material equivalence. Consider the following propositions

Pat smokes cigars. (1)
Winston smokes cigars. (2)

symbolized in PL as "P" and "W". Suppose that, in fact, both Pat and Winston do smoke cigars. Then (1) and (2) will be materially equivalent i.e. $P \equiv W$. If, as a matter of fact, neither smokes cigars then again $P \equiv W$ since once more P and W have the same truth value. However it is easy to imagine a possible world where just one of Pat and Winston smokes cigars: in such a world $P \equiv W$ will be false. So P is not necessarily equivalent to W i.e. $P \equiv W$ is not a necessary truth. In this text, whenever the term "equivalence" is used without qualification, necessary equivalence is intended.

Now look at the following propositions.

Pat and Winston smoke cigars. (3)
Winston and Pat smoke cigars. (4)

It should be obvious that (3) must have the same truth value as (4) in any possible world i.e. (3) is necessarily equivalent to (4). This may be demonstrated with the aid of a truth table. Using a common matrix, the truth table for (3) and (4) is as follows:

		(3)	(4)
P	W	$P \;\&\; W$	$W \;\&\; P$
1	1	1	1
1	0	0	0
0	1	0	0
0	0	0	0

Since the rows include all possible worlds, the matching between the main columns of

(3) and (4) shows they agree in truth value in each possible world. So (3) is equivalent to (4), and (4) is equivalent to (3) i.e. (3) and (4) are equivalent. In general, *given any truth table for two propositions, if their main columns match they are necessarily equivalent.*

The second modal relation we consider is that of contradictoriness. Given any propositions p and q, we say that:

> **p is *contradictory* to q iff p has the opposite truth value to q in all possible worlds**

i.e. p is contradictory to q iff there is no possible world where p and q are both true, and no possible world where p and q are both false (cf. §1.3). For instance (3) is contradictory to (5).

Either Pat doesn't smoke a cigar or Winston doesn't. (5)

This can be demonstrated by means of a truth table:

		(3)		(5)	
P	W	$P \& W$	$\sim P$	$\vee$	$\sim W$
1	1	1	0	0	0
1	0	0	0	1	1
0	1	0	1	1	0
0	0	0	1	1	1
		↑		↑	

Notice that on each row the main-column values of (3) and (5) are opposite. Since the rows include all possible worlds it follows that (3) is contradictory of (5), and that (5) is contradictory to (3) i.e. (3) and (5) are contradictories. In general, *given any truth table for two propositions, if their main-columns are opposite in value then they are contradictories.*

Our third modal relation is that of necessary implication. Given any propositions p and q, we say that:

> ***p necessarily implies q* iff there is no possible world with p true and q false.**

Necessary implication is much stronger than material implication: p necessarily implies q, iff $p \supset q$ is a necessary truth (not just true as a matter of fact). For example, if in fact Pat and Winston both smoke cigars then $P \supset W$. But since there is a possible world where Pat smokes cigars and Winston doesn't, P does not necessarily imply W. In this text, the term "implication" used without qualification will normally denote necessary implication.

If you look back at (3) and (1) it should be clear that (3) implies (1). We can demonstrate this by means of a truth table:

		(3)	(1)
P	W	$P \& W$	P
1	1	1	1
1	0	0	1
0	1	0	0
0	0	0	0

There is no row where (3) is true and (1) is false. Since the rows include all possible worlds it follows that (3) implies (1). In general, *given any truth table for two propositions, if there is no row where the first is true and the second is false then the first necessarily implies the second.*

Since there is a possible world where Pat smokes cigars and Winston doesn't (i.e. row 2 of the above table is possible), it is clear that even though (3) implies (1), (1) does not imply (3). So necessary implication is not a symmetric relation. A relation between p and q is *symmetric* iff it holds only if the converse relation (obtained by swapping p and q) also holds. Equivalence is symmetric because given any propositions p and q, if p is equivalent to q then q is equivalent to p. The relations of contradictoriness, inconsistency, contrariety, subcontrariety, indifference and consistency are also symmetric (this is easy to show from their definitions). Because implication is not symmetric however, we make separate mention of the relation of "converse implication" (or "being implied by"). Given any propositions p and q, we say that:

p* is *necessarily implied by q* iff *q* necessarily implies *p

The result that (3) implies (1) may be restated as "(1) is implied by (3)" or as "(1) follows from (3)".

Just as with the material relations $\supset$ and $\equiv$, if the modal relation of implication holds in both directions then the modal relation of equivalence holds. That is, *two propositions are necessarily equivalent iff each necessarily implies the other.* This is obvious from the definitions.

The two "paradoxes of material implication" mentioned in §2.4 also have their modal analogues. Consider the following propositions.

Pat smokes cigars and Pat doesn't smoke cigars. (6)
Canberra is the capital of Australia. (7)
Pat smokes cigars or Pat doesn't smoke cigars. (8)

Clearly (6) is a contradiction, (7) is a contingency, and (8) is a necessary truth. Symbolizing these and constructing a common truth table we obtain:

		(6)	(7)	(8)
P	C	$P \ \& \sim P$	C	$P \vee \sim P$
1	1	0	1	1
1	0	0	0	1
0	1	0	1	1
0	0	0	0	1

There is no row with (6) true and (7) false; so (6) necessarily implies (7). Moreover (6), or for that matter any contradiction, must imply any proposition, simply because there is no possible world where a contradiction is true (prove this for yourself using the definition of necessary implication).

Now look at the columns for (7) and (8). There is no row where (7) is true and (8) is false; so (7) implies (8). Moreover (8), or any necessary truth, will be implied by any proposition whatsoever, simply because there is no possible world where a necessary truth is false (prove this from the definition). So the following two curious results follow from the way we have defined necessary implication.

Any contradiction necessarily implies any proposition.

Any necessary truth is necessarily implied by any proposition.

These results will be considered again when we discuss validity of arguments.

The next modal relation, that of inconsistency, was discussed in §1.6. Given any propositions p and q, we say that:

***p* is *inconsistent* with *q* iff there is no possible world where both are true**

For example, (3) and (9) form an inconsistent pair.

Pat and Winston smoke cigars. (3)
Pat doesn't smoke cigars. (9)

This can be shown by means of a truth table:

		(3)	(9)
P	W	$P \& W$	$\sim P$
1	1	1	0
1	0	0	0
0	1	0	1
0	0	0	1

There is no row where both (3) and (9) are true. Since the rows include all possible worlds it follows that (3) is inconsistent with (9). In general, *given any truth table for two propositions, if there is no row where both are true then they are inconsistent with each other.*

EXERCISE 3.7A

For each of the following questions, use the dictionary below:

A = Alice is happy
B = Bernie is happy

1. Translate the propositions below into PL, then use a truth table to group them into four pairs of equivalent propositions.
   (a) Alice is happy.
   (b) If Alice is happy then Bernie is happy.
   (c) Either Alice or Bernie is happy.
   (d) If Bernie is happy then Alice is happy.
   (e) If Alice is not happy then Bernie is not happy.
   (f) Either Alice is happy or Alice is happy.
   (g) Either Bernie or Alice is happy.
   (h) If Bernie is not happy then Alice is not happy.
2. Use a truth table to group the following propositions into two pairs of contradictories.
   (a) Either Alice or Bernie is happy.
   (b) Alice is not happy but Bernie is.
   (c) Either Alice is happy or Bernie is not happy.
   (d) Alice is not happy and Bernie is not happy.
3. Use a truth table to show that (a) implies (b) and that (c) is implied by (d).
   (a) Alice is happy.
   (b) Either Alice or Bernie is happy.
   (c) Alice is not happy.
   (d) It's not the case that either Alice or Bernie is happy.
4. (a) Use a truth table to show that (i) and (ii) form an inconsistent pair.
      (i) If Alice is happy then Bernie is happy.
      (ii) Alice is happy but Bernie isn't.

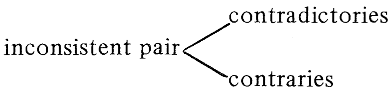
   (b) How many propositions are implied by the conjunction of (i) and (ii)?

(c) How many propositions imply the negation of the conjunction of (i) and (ii)?

The next modal relation, contrariety, was first met in § 1.3. Given any propositions p and q, we say that:

> ***p* is *contrary* to *q* iff there is no possible world where both are true but there is a possible world where both are false**

For example, (3) and (9) are contraries. If you look at the three definitions involved you will see that any inconsistent pair of propositions must fall into one of two distinct classes: contradictories or contraries.



Sometimes a truth table not only shows a pair of propositions is inconsistent but also reveals the type of inconsistency: for instance, the earlier truth table for (3) and (5) indicated they were contradictories. However, except in the rare case where both propositions are PL-contradictions, a standard truth table will not by itself reveal that two propositions are contraries. If you inspect the previous table for (3) and (9) you will notice that both propositions are false on row 2. But unless we know that row 2 is *possible*, we cannot justifiably conclude that there is a possible world where both (3) and (9) are false; and consequently we are unable to specify (3) and (9) as contraries. So to determine contraries we work with possible-truth tables. Looking at row 2 of the matrix we use our logical intuitions to determine that there is a possible world where Pat smokes cigars and Winston doesn't i.e. that row 2 is possible. Treating the table as a possible-truth table we are now able to say that (3) and (9) are contraries. In general, *given any possible-truth table for two propositions, they are contraries iff there is no row with both true but there is a row with both false.*

The next two modal relations, subcontrariety and indifference, have not been discussed before and are of somewhat lesser importance. Given any propositions p and q, we say that:

> ***p* is *subcontrary* to *q* iff there is no possible world where both are false but there is a possible world where both are true**

The following propositions form a pair of subcontraries:

Pluto is small or inhabited. (10)
Pluto is small or uninhabited. (11)

Symbolizing these and constructing a common truth table we get:

		(10)	(11)	
S	I	$S \lor I$	$S \lor$	$\sim I$
1	1	1	1	0
1	0	1	1	1
0	1	1	0	0
0	0	0	1	1
		↑	↑	

Since the rows include all possible worlds, the absence of any row with both (10) and (11) false reveals that there is no possible world where both are false. On rows 1 and 2 both propositions are true, but unless at least one of these rows is possible it does not follow that there is a possible world where both (10) and (11) are true. Hence the standard truth table fails to show that (10) and (11) are subcontraries. Once more, we resort to the possible-truth table method. We can easily imagine possible worlds where Pluto is small and inhabited (row 1), and adding that row 1 is possible allows us to deduce that (10) and (11) are subcontraries. We could have used row 2 instead, as it is also possible.

Except in the rare case where both propositions are tautologies, possible-truth tables must be used instead of standard truth tables to detect subcontraries. In general, *given any possible-truth table for two propositions, they are subcontraries iff there is no row with both false but there is a row with both true.*

Truth-functional indifference is defined as follows. Given any propositions p and q,

p is *indifferent* to q iff there is a possible world where both are true, another where both are false, another with p true and q false, and another with p false and q true

To take an easy example, propositions (1) and (2) are indifferent.

Pat smokes cigars. (1)
Winston smokes cigars. (2)

It should be obvious that a standard truth table can never detect indifference: a possible-truth table is required. In general, *given any possible-truth table for two propositions, they are indifferent iff there are rows for all four truth-value permutations of the propositions.* Because all permutations are possible with indifferent propositions, we can never deduce the truth value of one from the truth value of the other. Although two propositions may thus be truth-functionally indifferent, they may still be connected in other ways (the *probability* of one may depend on the truth value of the other: to pursue this point however would take us into the area of inductive logic).

The final modal relation we consider is that of consistency. This was introduced in §1.6 and may be defined as follows. Given any propositions p and q,

p is *consistent* with q iff there is a possible world where both are true

For example, (1) and (2) are consistent, and so are (10) and (11). It should be clear from the definitions that "inconsistent" means "not consistent". Except in the rare case where both propositions are tautologies, possible-truth tables rather than standard truth tables are required to detect consistency. For instance, in the previous table for (10) and (11) we need to know that at least one of rows 1 and 2 is possible in order to pronounce (10) and (11) as a consistent pair. In general, *given any possible-truth table for two propositions, they are consistent iff there is a row with both true.*

Let's return now to the first five modal relations discussed. For each of these we found that the relation holds *if* a certain pattern occurs in the standard truth table for the propositions. Does the relation hold *only if* the pattern occurs? The answer is "No". Consider for example the following two propositions.

The Earth has exactly seven moons. (12)
The Earth does not have eight moons. (13)

Using the dictionary

S = The Earth has exactly seven moons
E = The Earth has eight moons

we obtain the following truth table:

		(12)	(13)
S	E	S	$\sim E$
1	1	1	0
1	0	1	1
0	1	0	0
0	0	0	1

Since none of the patterns discussed for the standard truth table tests occurs here, the relation between (12) and (13) is *indeterminate* within PC. By going beyond PC in constructing a possible-truth table however, the relation can be found. Using our intuitions on the matrix rows we find that just row 1 is impossible, thus obtaining the possible-truth table:

			(12)	(13)
	S	E	S	$\sim E$
×	1	1	1	0
	1	0	1	1
	0	1	0	0
	0	0	0	1

Looking at the three remaining rows we see that there is no possible world where (12) is true and (13) is false i.e. (12) necessarily implies (13).

Similarly, the other relations may fail to be detected by standard truth tables when the matrix propositions themselves are not indifferent to each other. Standard truth table patterns may thus be sufficient but are not necessary for the modal relations to hold. With possible-truth tables a set of necessary and sufficient conditions can be given i.e. an "iff" test can be applied rather than merely an "if" test. We may summarise these tests for our nine modal relations as follows.

Given a *possible-truth table* for p and q:

1. p is *equivalent* to q	iff their main columns match
2. p is *contradictory* to q	iff their main columns are opposite.
3. p is *contrary* to q	iff there is no row with both true but there is a row with both false
4. p is *subcontrary* to q	iff there is no row with both false but there is a row with both true
5. p *implies* q	iff there is no row with p true and q false
6. p is *implied by* q	iff there is no row with q true and p false
7. p is *indifferent to* q	iff there are rows for all four truth-value permutations
8. p is *consistent* with q	iff there is a row with both true
9. p is *inconsistent* with q	iff there is no row with both true

The last two relations above are collectively exhaustive (i.e. any pair of propositions must exhibit one of these relations) and mutually exclusive (no pair can exhibit more than one of these relations). Since they are less specific than the other relations we will say no more about them at this stage.

The first seven relations above are collectively exhaustive but are not mutually exclusive. For example, if p is equivalent to q, then p both implies and is implied by q: in this case it is better to state the relationship between p and q as one of equivalence rather than implication since the former description provides more information about the relation. Given two propositions, if we want to zero-in efficiently on the most informative relation obtaining between them it is best to test in the order listed above, stopping as soon as we find a relation that holds between them. *If* we find the propositions are not contradictories *then* they are contraries iff there is no row with them both true and they are subcontraries iff there is no row with them both false (explain why these half-tests for contraries and subcontraries are adequate here). Because the seven relations are exhaustive, if we find the first six relations do not hold we know immediately that the propositions must be indifferent. This method may be summarised as follows.

Method: Compute the main columns of *p* and *q* using a common matrix

Eliminate any impossible rows.

Test in the following order, stopping as soon as a relation is found by the tests shown in parenthesis:

1.	***p* is equivalent to *q***	**(matching main columns)**
2.	***p* is contradictory to *q***	**(opposite main columns)**
3.	***p* is contrary to *q***	**(no row with both true)**
4.	***p* is subcontrary to *q***	**(no row with both false)**
5.	***p* implies *q***	**(no row with *p* true and *q* false)**
6.	***p* is implied by *q***	**(no row with *q* true and *p* false)**
7.	***p* is indifferent to *q***	

It will be convenient to extend the notion of indifference to any number of propositions as follows. The propositions $p_1, p_2, \ldots, p_n$ are *indifferent* iff all permutations of truth-values for the propositions are possible. *When the matrix propositions of a table are indifferent, the possible-truth table is identical to the standard truth table* since all rows are possible.

Example: Given that *A*, *B* and *C* are indifferent propositions, discuss the modal relationship between (a) and (b)

(a) $A \supset (B \supset C)$
(b) $(A \supset B) \supset C$

			(a)		(b)	
A	B	C	$A \supset$	$(B \supset C)$	$(A \supset B)$	$\supset C$
1	1	1	1	1	1	1
1	1	0	0	0	1	0
1	0	1	1	1	0	1
1	0	0	1	1	0	1
0	1	1	1	1	1	1
0	1	0	1	0	1	0
0	0	1	1	1	1	1
0	0	0	1	1	1	0
			↑			↑

Since *A*, *B* and *C* are indifferent, this is a possible-truth table.

(a) and (b) are not equivalent	(see e.g., row 6)
(a) and (b) are not contradictories	(see e.g., row 1)
(a) and (b) are not contraries	(see e.g., row 1)
(a) and (b) are not subcontraries	(see row 2)
(a) does not imply (b)	(see e.g., row 6)

There is no row where (b) is true and (a) is false.

∴ (b) implies (a)

i.e. (a) is implied by (b).

Comments: The justification provided below the table for the answer would normally be done mentally rather than written down. With problems like this it is very helpful to use a different colour for the main columns to highlight them for comparison.

NOTES

The following table summarizes our modal relations by listing which main-column combinations are allowed, forbidden or mandatory in a possible-truth table.

	Allowed		*Forbidden*		*Mandatory*	
	p	q	p	q	p	q
p is equivalent to q	1	1	1	0		
	0	0	0	1		
p is contradictory to q	1	0	1	1		
	0	1	0	0		
p is contrary to q	1	0	1	1	0	0
	0	1				
	0	0				
p is subcontrary to q	1	1	0	0	1	1
	1	0				
	0	1				
p implies q	1	1	1	0		
	0	1				
	0	0				
p is implied by q	1	1	0	1		
	1	0				
	0	0				
p is indifferent to q	1	1			1	1
	1	0			1	0
	0	1			0	1
	0	0			0	0
p is consistent with q	1	1			1	1
	1	0				
	0	1				
	0	0				
p is inconsistent with q	1	0	1	1		
	0	1				
	0	0				

(A blank in the mandatory column indicates that while no specific combination is mandatory, at least one of the allowed combinations will of course be present.)

A "reduced truth table" for a pair of propositions may be obtained by first determining their possible-truth table and then eliminating rows where main-column value-pairs are merely a repeat of a previous row. Reduced truth tables provide a quick way of checking for the 15 possible modal relations that may exist between two propositions. For further details consult Bradley and Swartz (*op. cit.*).

Although the negation is the most common contradictory of a proposition, it can easily be shown that each proposition has an infinite number of propositions which are contradictory to it. First note that there is an infinite number of necessary truths (e.g., $0 < 1, 0 < 2, 0 < 3, \ldots$). Also note that any proposition p will have $\sim p$ & T as a contradictory, where T is any necessary truth. (As an exercise, prove this for yourself.) Since there are infinitely many T's this completes the proof.

Some authors prefer to use the term "independence" for truth-functional indifference. We have avoided this practice because in inductive logic the term "independence" standardly denotes a much stronger lack of connection: there p and q are said to be independent iff the *probability* of p is unrelated to the truth value of q (i.e. the probability of p given q = the probability of p).

EXERCISE 3.7B

Key: When asked to determine the "first" modal relation, indicate the first that holds from the below list, by selecting the appropriate key number.

1. equivalent
2. contradictory
3. contrary
4. subcontrary
5. first implies second
6. first implied by second
7. indifferent

1. The main-columns of a possible-truth table for propositions (a) – (h) are shown below.

(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
1	0	1	0	1	0	1	1
1	0	1	0	0	1	1	0
0	1	0	0	1	0	0	0
0	1	0	0	1	1	1	0

Determine the first modal relation between the following.

(i) (a) and (b)
(ii) (a) and (c)
(iii) (a) and (d)
(iv) (a) and (e)
(v) (a) and (f)
(vi) (a) and (g)
(vii) (a) and (h)
(viii) (b) and (d)
(ix) (b) and (e)
(x) (b) and (f)
(xi) (b) and (g)
(xii) (b) and (h)
(xiii) (e) and (f)
(xiv) (e) and (h)
(xv) (f) and (g)
(xvi) (g) and (h)

2. Four propositions are defined in terms of two indifferent propositions A and B as follows.

(a) $A \& B$
(b) $(A \supset B) \supset A$
(c) $A \supset (B \supset A)$
(d) $(B \supset A) \supset B$

Construct a common truth table for these four propositions, then state TRUE or FALSE for each of the following.

(i) (a) is equivalent to (b)
(ii) (a) implies (b)
(iii) (a) is implied by (b)
(iv) (b) is inconsistent with (d)
(v) (c) is implied by (b)
(vi) (c) is subcontrary to (d)
(vii) (c) is contrary to (b)

(viii) (b) is indifferent to (d)
(ix) (c) is implied by each of the others

3. Five propositions are defined in terms of three indifferent propositions A, B and C as follows.

(a) $A \supset (B \vee C)$
(b) $A \equiv (B \vee C)$
(c) $(\sim B \& \sim C) \supset \sim A$
(d) $A \& \sim (B \vee C)$
(e) $A \supset C$

Construct a common truth table then determine the first modal relation between the following. (Use the Key)

(i) (a) and (b)
(ii) (a) and (c)
(iii) (a) and (d)
(iv) (a) and (e)
(v) (b) and (e)

4. Five propositions are defined in terms of three indifferent propositions A, B and C as follows.

(a) $((C \supset A) \supset B)$
(b) $(\sim (B \vee C) \vee (\sim B \& A))$
(c) $(B \vee (\sim A \& C))$
(d) $(A \not\equiv B)$
(e) $((A \equiv C) \vee \sim C)$

Use tables to determine the first modal relation between the following.

(i) (a) and (b)
(ii) (b) and (e)
(iii) (c) and (d)
(iv) (d) and (e)
(v) (e) and (a)

5. Translate the following propositions into PL using the suggested dictionary and interpreting the English operators straightforwardly in terms of PL-operators.

T = You tickle me
L = I laugh
U = I laugh uncontrollably

(a) If you tickle me I laugh.
(b) I laugh uncontrollably if you tickle me.
(c) If I'm not laughing then you're not tickling me.
(d) Although you're tickling me I'm not laughing.
(e) If you tickle me I laugh but not uncontrollably.

Now use possible-truth tables to determine the first modal relation between the following. (Use the Key)

(i) (a) and (b)
(ii) (a) and (c)
(iii) (a) and (d)
(iv) (b) and (d)
(v) (e) and (a)
(vi) (b) and (e)

6. Classify each of the pairs mentioned in Question 5 as consistent or inconsistent.

7. Translate the following pairs into PL using the suggested dictionary, and then use a possible-truth table to classify them as either contradictories or contraries.

   (a) The chairman is both honest and competent.
   The chairman is neither honest nor competent.

   (b) The chairman is either not honest or not competent.
   The chairman is competent and honest.

   (c) The chairman is competent but dishonest.
   The chairman is honest but incompetent.

*8. Paul and David and engaged in a debate about Fred's pets. They make the following claims:

Paul:	Although Fred has a large dog he doesn't have a cat.
David:	Fred has neither a dog nor a cat.

Using the suggested dictionary, construct a possible-truth table for these claims then use it to answer the following questions.

C = Fred has a cat
D = Fred has a dog
L = Fred has a large dog

(a) Have Paul and David adopted contradictory positions?
(b) Have Paul and David adopted contrary positions?
(c) Describe four different possible worlds in which both Paul and David have made false claims.
(d) Describe two different possible worlds in which just one of Paul and David is correct.
(e) If it is established that Paul is wrong, does this imply that David is right?
(f) If it is established that Paul is right, does this imply that David is wrong?

*9. Selena and Linda are also discussing the matter of Fred's pets. They take the following positions:

Selena:	Fred has a cat but he doesn't have a large dog.
Linda:	Either Fred does have a dog or he doesn't have a cat.

Using the same dictionary as for the previous question, construct a possible-truth table for these claims then use it to answer the following questions.

(a) Have Selena and Linda adopted indifferent positions?
(b) Have they adopted subcontrary positions?
(c) Describe a possible world (if any) where both are correct.
(d) Describe the different possible worlds where just one is correct.
(e) If it is established that Selena is wrong does this imply that Linda is right?
(f) If it is established that Selena is right does this imply that Linda is wrong?

10. Answer TRUE or FALSE for each of the following.
Given any propositions p and q:

(a) if p is equivalent to q then p implies q
(b) if p implies q and q implies p then p and q are equivalent
(c) if p and q are inconsistent they are contradictories
(d) if p is consistent with q then q is consistent with p
(e) if p implies q then q implies p
(f) p implies any contradiction
(g) p implies any necessary truth
(h) p is implied by any contradiction

(i) p is implied by any necessary truth
(j) if p and q are contraries then they are inconsistent
(k) if p and q are contraries they are both false in some possible world
*(l) if p and q are equivalent they are not contraries
*(m) if p and q are equivalent they are not subcontraries
*(n) if p and q are contradictories then p does not imply q
(o) if p and q are indifferent they are consistent

*11. After a course in philosophical scepticism, a student was heard to remark:

> There's one thing I'm sure about now.
> And that is that I can't certain of anything.

Discuss whether or not the student was consistent in his remarks.

3.8 SOME IMPORTANT TAUTOLOGIES

Recall that a well formed formula of PL is a tautology iff its value = 1 for all assignments of values to its propositional letters. In this section we look at a few of those tautologies which attract a special name because of their fundamental logical role. Some of these have been met before and proved to be tautologous; the proof of the others is left as a truth table exercise.

Let's begin with the following tautology.

$$p \vee \sim p \tag{1}$$

If we uniformly replace the propositional variable p with the form $(p \ \& \ q)$ we obtain:

$$(p \ \& \ q) \vee \sim(p \ \& \ q) \tag{2}$$

Will (2) also be a tautology? A quick test with a table will show it is. In fact, if you think about it, uniformly replacing p in (1) with any PL-wff will result in a tautology. This follows from the fact that a wff can take on no more truth values than a propositional variable can (viz. 1 and 0). So the following two examples will also be tautologies:

$$\sim p \vee \sim\sim p \tag{3}$$

$$(q \ \& \ (r \vee s)) \vee \sim(q \ \& \ (r \vee s)) \tag{4}$$

These were obtained from (1) by uniformly replacing p with $\sim p$ and $(q \ \& \ (r \vee s))$ respectively. In general, given any PL-wff α, the following will be a tautology:

$$\alpha \vee \sim \alpha \tag{5}$$

This kind of result can be generalized. Consider for instance the following tautology.

$$p \ \& \ q \ . \equiv . \ q \ \& \ p \tag{6}$$

Here we have used dots to highlight the main operator. If we unformly replace p with $(p \supset q)$ and q with $\sim r$ we obtain:

$$(p \supset q) \ \& \sim r \ . \equiv . \sim r \ \& \ (p \supset q) \tag{7}$$

If you test (7) you will find that it is also a tautology. It is not hard to see that if we replace the p and q in (6) with any PL-wffs α and β we must end up with a tautology i.e. any wff of the following form is tautologous:

$$\alpha \ \& \ \beta \ . \equiv . \ \beta \ \& \ \alpha \tag{8}$$

The following general result should now be obvious. *If the propositional letters in any tautologous formula are uniformly replaced with PL-wffs then the resulting formula is also a tautology.* While we will typically use propositional variables (p, q, ...) instead of

PL-wff variables (α, β, ...) when identifying important tautologies, it should be borne in mind that the generalization above holds. This is important in later work.

In §3.5 we noted that any proposition with a tautologous form is itself a tautology, and moreover a necessary truth. You may recall the next two examples and their symbolizations:

Either reincarnation occurs or it doesn't. (9)
$R \vee \sim R$ (9a)
Alice and Bernie are both happy or they're not both happy. (10)
$(A \ \& \ B) \vee \sim(A \ \& \ B)$ (10a)

Since both (9) and (10) have the tautologous form "$p \vee \sim p$", they are tautologies. In general, given any proposition p, $p \vee \sim p$ is a necessary truth i.e. any proposition must be either true or not true: this result is called the *Law of Excluded Middle* (LEM) since it excludes any middle state between true and not true. Sometimes the name "Law of Excluded Middle" is applied to PL-forms, sometimes to PL-sentences and sometimes to propositions. Thus, letting "wf" abbreviate "well formed", the label "LEM" might denote any of the following three results:

1. the form "$p \vee \sim p$" is a tautology, as is any wf PL-form $\alpha \vee \sim\alpha$;
2. any wf PL-sentence of the form $p \vee \sim p$ is a tautology;
3. any proposition of the form $p \vee \sim p$ is a tautology.

To simplify matters we introduce the notation "T:" as an abbreviation for "This is a tautology:" and take it that the formal, sentential and propositional versions of a tautology may all be expressed by listing the tautologous form. So the three aspects of LEM mentioned above may be expressed concisely as follows.

T: $p \vee \sim p$ **(LEM)**

With this understood, let's look at some more logical laws. The *Law of Non-Contradiction* (LNC) states that no proposition can be both true and not true i.e.

T: $\sim(p \ \& \sim p)$ **(LNC)**

Both LEM and LNC may be combined into a single law which we call the *Law of Bi-Valence* (LBV).

T: $p \not\equiv \sim p$ **(LBV)**

i.e. any proposition must be true or not true but not both. If we agree that "false" means "not true" then this law says that each proposition has exactly one of the values "true" and "false". Because classical logic limits the number of possible truth values to these two, it is sometimes called a "*2-valued logic*": hence the term "bi-valence".

In §3.7 we discussed various modal relations between propositions. Some of these relations (e.g., contradictoriness, contrariety, subcontrariety) are of primary importance in dialogue situations (e.g., in indicating clearly what bearing the truth or falsity of one position in a two-party debate has on the other position). Some other relations, particularly equivalence and implication, are of primary importance in the construction and evaluation of arguments and logical deductions: because the analysis of such deductive reasoning is facilitated by reference to the underlying forms, it will be helpful to introduce the notions of equivalence and implication between *formulae*, not just propositions.

Given any two PL-wffs α and β we say that α and β are *tautologically equivalent* iff $\alpha \equiv \beta$ is a tautology. So two formulae will be tautologically equivalent *iff they have matching main columns* in their truth table. For example, the form p is tautologically equivalent to ${\sim}{\sim}p$. This can be established from the table below either by noting that $p \equiv {\sim}{\sim}p$ is a tautology, or, more quickly, by noting that p and ${\sim}{\sim}p$ have matching columns.

p	p	${\sim}{\sim}p$	$p \equiv {\sim}{\sim}p$
1	1	1 0	1
0	0	0 1	1
	↑	↑	↑

Because we often use PL-sentences to denote specific propositions, it will be convenient to extend the notion of tautological equivalence to propositions. Let us agree that *two propositions are tautologically equivalent iff they have tautologically equivalent forms.* If a proposition has a tautologous form, it is a necessary truth. Hence *if two propositions are tautologically equivalent then they are necessarily equivalent.* From our work in the previous section it should be clear that the converse of this general result does not hold e.g., the propositions "Some doors are open" and "Not all doors are unopen" are necessarily equivalent but not tautologically equivalent.

Look at propositions (11) and (12): as their PL-symbolizations (11a) and (12a) make clear, they have the tautologically equivalent forms p and ${\sim}{\sim}p$. So (11) and (12) are tautologically equivalent, and hence necessarily equivalent.

It's Monday. (11)
M (11a)

It's not the case that it's not Monday. (12)
${\sim}{\sim}M$ (12a)

The construction exemplified by (12) is usually called a "double negative" in English grammar. In logic, the general result that any proposition is equivalent to the negation of its negation is known as the law of *Double Negation* (DN). This may be expressed as follows.

T: $p \equiv {\sim}{\sim}p$ **(DN)**

In §2.4, both commutative and associative laws were discussed. These may now be formalized. Each of &, $\vee$, $\equiv$ and $\not\equiv$ are *commutative* i.e.

T: $p \,\&\, q \;.\!\equiv\!.\; q \,\&\, p$ **(Com &)**
T: $p \vee q \;.\!\equiv\!.\; q \vee p$ **(Com $\vee$)**
T: $p \equiv q \;.\!\equiv\!.\; q \equiv p$ **(Com $\equiv$)**
T: $p \not\equiv q \;.\!\equiv\!.\; q \not\equiv p$ **(Com $\not\equiv$)**

These operators are also *associative* i.e.

T: $p \,\&\, (q \,\&\, r) \;.\!\equiv\!.\; (p \,\&\, q) \,\&\, r$ **(Assoc &)**
T: $p \vee (q \vee r) \;.\!\equiv\!.\; (p \vee q) \vee r$ **(Assoc $\vee$)**
T: $p \equiv (q \equiv r) \;.\!\equiv\!.\; (p \equiv q) \equiv r$ **(Assoc $\equiv$)**
T: $p \not\equiv (q \not\equiv r) \;.\!\equiv\!.\; (p \not\equiv q) \not\equiv r$ **(Assoc $\not\equiv$)**

Note that $\supset$ is neither commutative nor associative.

The next two laws have been alluded to in earlier discussions about equivalent translations. They are called *De Morgan's Laws* (DeM) after the famous mathematician Augustus De Morgan.

T: $\sim(p \,\&\, q) \;.\!\equiv.\; \sim p \vee \sim q$ **(DeM)**
T: $\sim(p \vee q) \;.\!\equiv.\; \sim p \,\&\, \sim q$ **(DeM)**

It is interesting to compare $\sim$ with the unary - of mathematics. The algebraic result $x = --x$ is similar to DN, but this is as far as the analogy goes. For instance, while unary - distributes over +, i.e. $-(x + y) = -x + -y$, any attempt to distribute $\sim$ over & or $\vee$ results in a conversion between & and $\vee$ as shown by DeM.

As preparation for our next equivalence, use your logical intuitions to determine which, if any, of (14) – (16) are equivalent to (13).

If Smith is a woman then Smith is human. (13)
If Smith is human then Smith is a woman. (14)
If Smith is not a woman then Smith is not human. (15)
If Smith is not human then Smith is not a woman. (16)

Notice that (14), the *converse* of (13), is obtained by swapping the antecedent and consequent; (15), the *inverse* of (13), is obtained by negating the antecedent and consequent where they stand; (16), the *contrapositive* of (13), is obtained by both swapping and negating the antecedent and consequent. Did you see that (16) is equivalent to (13)? As regards (14) and (15), these are equivalent to each other, but not to (13). Provided "$\supset$" is an acceptable translation for "if ... then" these results can easily be demonstrated with a truth table (try this for yourself). The terms above are also used in reference to $\supset$-conditionals. Thus $p \supset q$ has $q \supset p$ as its converse, $\sim p \supset \sim q$ as its inverse and $\sim q \supset \sim p$ as its contrapositive. The equivalence between a proposition or formula and its contrapositive is known as *Contraposition* (Contrap).

T: $p \supset q \;.\!\equiv.\; \sim q \supset \sim p$ **(Contrap)**

Too frequently, conditional statements are made in the hope that the listener will erroneously assume the inverse e.g.,

If you elect me, conditions will improve. (17)
If you use Goodo toothpaste, your teeth will sparkle. (18)

Their (non-equivalent) inverses are:

If you don't elect me, conditions won't improve. (19)
If you don't use Goodo toothpaste, your teeth won't sparkle. (20)

Before discussing our last equivalence for this section, it will be timely to introduce the notion of implication between formulae. Given any PL-wffs α and β, we say that α *tautologically implies* β iff $\alpha \supset \beta$ is a tautology i.e. α tautologically implies β *iff there is no truth table row where* $\alpha = 1$ *and* $\beta = 0$. For example, the table below shows that p tautologically implies $p \vee q$ but not vice versa.

p	q	p	$p \vee q$
1	1	1	1
1	0	1	1
0	1	0	1
0	0	0	0

As with tautological equivalence, we extend this notion to propositions. We say that *one proposition tautologically implies a second proposition iff the first has a form which tautologically implies a form of the second.* For example, (21) tautologically implies (22) since they have the forms p and $p \vee q$.

It's Monday. (21)
It's Monday or Tuesday. (22)

Because of the fact that any proposition with a tautologous form is a necessary truth, it follows that *if one proposition tautologically implies a second proposition then the first proposition necessarily implies the second.* From § 3.7 it is obvious that the converse of this general result does not hold e.g., "Terry is 35" necessarily implies, but does not tautologically imply, the proposition "Terry is older than 30".

Look now at propositions (23) and (24). Intuitively, would you say that (23) implies (24)?

If you are a human adult and a male then you are a man. (23)
If you are a human adult then if you are a male you are a man. (24)

It does imply (24). If we agree that our PL-operators provide adequate translations, this can easily be shown with a truth table. This example is an instance of the law of *Exportation* (Exp) i.e.

T: $(p \,\&\, q) \supset r \;.\!\supset\!.\; p \supset (q \supset r)$ **(Exp)**

This says that $(p \,\&\, q) \supset r$ tautologically implies $p \supset (q \supset r)$. It is called "Exportation" since the q is "exported" from the antecedent $(p \,\&\, q)$. Here now is another question. Does (24) imply (23)?

Again, a truth table will readily show the answer is "Yes". This is a case of the law of *Importation* (Imp) i.e.

T: $p \supset (q \supset r) \;.\!\supset\!.\; (p \,\&\, q) \supset r$ **(Imp)**

Here the q is "imported" back into the antecedent $(p \,\&\, q)$. From our definitions it will be evident that *tautological equivalence occurs iff there is tautological implication in both directions.* So Exp and Imp may be combined to form the following equivalence which we will call *Export- Import* (Exim)

T: $(p \,\&\, q) \supset r \;.\!\equiv\!.\; p \supset (q \supset r)$ **(Exim)**

Well that's enough tautologies for the moment. We will run across several further equivalences and implications later on. In order to abbreviate our reference to such results it will be convenient to introduce a few more symbols. The symbol "$\Leftrightarrow$" will be used to denote "system equivalence" i.e. modal equivalence that can be established within the logical system being discussed. The system we are discussing at the moment is Propositional Calculus: in this case $\Leftrightarrow$ denotes tautological equivalence. For example, instead of expressing DN as

T: $\sim\sim p \equiv p$

we could write this more briefly as

$$\sim\sim p \Leftrightarrow p$$

Notice that the "T:" is no longer required. To minimize the use of brackets or dots and to emphasize that $\Leftrightarrow$ is a modal rather than a truth-functional operator we give $\Leftrightarrow$ a lower priority than our truth-functional operators. For instance when Contraposition is stated as below, $\Leftrightarrow$ must be read as the main operator.

$$p \supset q \;\Leftrightarrow\; \sim q \supset \sim p$$

In a similar manner, we use "$\Rightarrow$" and "$\Leftarrow$" to denote "system implication" in the directions indicated by the arrow heads. Within PC "$\Rightarrow$" and "$\Leftarrow$" may be read as "tautologically implies" and "is tautologically implied by" respectively. As with $\Leftrightarrow$, $\Rightarrow$ and $\Leftarrow$ are modal operators and are assigned lower priority than truth-functional operators. For example, Exportation and Importation may be stated more briefly as:

$$(p \,\&\, q) \supset r \;\Rightarrow\; p \supset (q \supset r)$$
$$(p \,\&\, q) \supset r \;\Leftarrow\; p \supset (q \supset r)$$

Obviously, $\Leftrightarrow$ amounts to a conjunction of $\Rightarrow$ and $\Leftarrow$. These symbols may be used with both formulae and propositions. When propositions are involved, the corresponding necessary relation also holds. For instance, in reference to propositions (11) and (12) the result $\sim\sim M \Leftrightarrow M$ asserts tautological equivalence (and consequently necessary equivalence) between (11) and (12).

NOTES

Though nine modal relations were defined between propositions, we have defined only three for formulae. Other relations could be defined for formulae but there is little point in doing so.

We have gone to some pains in distinguishing between tautological and necessary relations, partly to indicate the limitations of PC and partly to prevent use/mention ambiguities arising with forms.

Contraposition is sometimes called "Transposition". Often the Exp–Imp equivalence is called simply "Exportation".

Unlike $\sim$, $\&$, $\vee$, $\supset$, $\equiv$ and $\not\equiv$, the modal operators $\Leftrightarrow$, $\Rightarrow$ and $\Leftarrow$ are not truth-functional. For instance, if two propositions p and q are true this fixes the value of $p \equiv q$ as true; but the value of $p \Leftrightarrow q$ is not fixed until we know how the values of p and q are distributed across possible worlds.

If when working in a system other than PC (e.g., Quantification Theory) it is desired to emphasize that a certain equivalence of implication is tautological, we can add the subscript "T" to the system relation symbols, thus: $\underset{T}{\Leftrightarrow}$, $\underset{T}{\Rightarrow}$, $\underset{T}{\Leftarrow}$.

EXERCISE 3.8

1 Each of the following formulae is a tautology: name the law of which it is an instance.

(a) $\sim(q \,\&\, \sim q)$
(b) $(q \supset r) \vee \sim(q \supset r)$
(c) $\sim p \,\&\, q \;.\!\equiv\!.\; q \,\&\, \sim p$
(d) $(p \equiv q) \not\equiv \sim(p \equiv q)$
(e) $\sim(q \,\&\, (r \supset s)) \;.\!\equiv\!.\; \sim q \vee \sim(r \supset s)$
(f) $(q \,\&\, r) \supset \sim s \;.\!\equiv\!.\; \sim\sim s \supset \sim(q \,\&\, r)$
(g) $(p \vee \sim q) \;.\!\equiv\!.\; \sim\sim(p \vee \sim q)$
(h) $\sim((p \,\&\, q) \vee r) \;.\!\equiv\!.\; \sim(p \,\&\, q) \,\&\, \sim r$
(i) $\sim p \vee (r \vee (q \,\&\, s)) \;.\!\equiv\!.\; (\sim p \vee r) \vee (q \,\&\, s)$
(j) $(p \,\&\, (q \vee r)) \supset p \;.\!\equiv\!.\; p \supset ((q \vee r) \supset p)$

2. (a) Discuss whether or not the sense of "and" in the following examples is commutative.
   (i) She tickled me and I laughed.
   (ii) Meditate and you will find peace.

   (b) Do the following examples show that "and" is not associative?
   (i) My two exams are on genetics, and probability and statistics.
   (ii) My two exams are on genetics and probability, and statistics.

3 Write down in order the converse, inverse and contrapositive of the following formula, using DN where relevant to simplify your answer.

$$\sim(p \vee q) \supset (\sim p \,\&\, \sim q)$$

4. Which of the following is equivalent to "If you have studied you will pass"?
   - (a) If you haven't studied you won't pass.
   - (b) If you pass then you have studied.
   - (c) If you don't pass you haven't studied.

5. Which of the following are equivalent to
   "If it's not good meat, it's not Kilcoy meat."?

   (Hint: First translate the above proposition into a simpler equivalent. Translation into PL will help for some, but not all, of the examples.)

   - (a) If it's good meat then it's Kilcoy meat.
   - (b) It's Kilcoy meat if it's good meat.
   - (c) It's not Kilcoy meat only if it's not good meat.
   - (d) If it's Kilcoy meat, it's good meat.
   - (e) If it's not Kilcoy meat it's not good meat.
   - (f) It's not Kilcoy meat if it's not good meat.
   - (g) It's good meat only if it's Kilcoy meat.
   - (h) It's Kilcoy meat if only it's good meat.
   - (i) It's good meat if and only if it's Kilcoy meat.
   - (j) Being Kilcoy meat is necessary for being good meat.
   - (k) For it to be good meat it is sufficient that it be Kilcoy meat.
   - (l) Only Kilcoy meat is good meat.
   - (m) Kilcoy meat is good meat.
   - (n) Only good meat is Kilcoy meat.
   - (o) Kilcoy meat is the only good meat.
   - (p) Good meat is only Kilcoy meat.
   - (q) Kilcoy meat only is good meat.

3.9 SUMMARY

A *propositional letter* may be either a propositional variable or a propositional constant. Wffs of PL are evaluated in assembly-line order i.e. by the following *priority convention:*

$$\left\downarrow \begin{array}{l} p,\ q,\ \ldots \\ (\ ) \\ \sim \\ \&,\ \vee,\ \supset,\ \equiv\ ,\ \not\equiv \end{array}\right.$$

In a formula's truth table, columns of values are placed under the evaluated symbols, the *main column* (i.e. the last calculated) being identified with an arrow: this column gives the values of the formula as a whole. A formula with n propositional letters has 2^n rows in its truth table.

PL-form	*tautology*	(main-column values: all 1)
	contradiction	(main-column values: all 0)
	contingency	(main-column values: some 1, some 0)

The following one-operand evaluation rules save work:

$$\begin{array}{llll} 0\ \&\ \ldots = 0 & & \ldots\ \&\ 0 = 0 \\ 1 \vee \ldots = 1 & & \ldots \vee 1 = 1 \\ 0 \supset \ldots = 1 & & \ldots \supset 1 = 1 \end{array}$$

Propositions	*Necessary Truth*	(true in all possible worlds)
	Contradiction	(false in all possible worlds)
	Contingency	(true in just some possible worlds)

The rows of a *proposition's* truth table collectively cover all possible worlds, though some rows may not be possible. If a proposition has a tautologous form it is a *tautology* (and necessary truth); if it has a self-contradictory PL-form it is a *PC-Contradiction;* in all other cases it is *PC-Indeterminate*, and the main-column of its explicit truth table contains a mixture of 1's and 0's.

Possible-truth tables are truth tables where each row is possible. They are formed from standard truth tables by eliminating each row that has an impossible matrix permutation. Given a possible-truth table, any proposition may be classified as a necessary truth, contradiction or contingency according as its main-column values are all true, all false or a mixture respectively.

Using "pw" as an abbreviation for "possible world", nine modal relations for propositions *p, q* may be defined as follows.

1. *p* is *necessarily equivalent* to *q* iff *p* has the same truth value as *q* in all pws.
2. *p* is *contradictory* to *q* iff *p* has the opposite value to *q* in all pws.
3. *p* is *contrary* to *q* iff there is no pw with both true but there is a pw with both false.
4. *p* is *subcontrary* to *q* iff there is no pw with both false but there is a pw with both true.
5. *p necessarily implies q* iff there is no pw with *p* true and *q* false.
6. *p* is *necessarily implied* by *q* iff *q* necessarily implies *p*
7. *p* is *indifferent* to *q* iff pws exist for each of the 4 truth value permutations.
8. *p* is *consistent* with *q* iff there is a pw where both are true.
9. *p* is *inconsistent* with *q* iff there is no pw where both are true.

Except for 5 and 6, each of these relations is *symmetric* (if the relations holds between *p* and *q* (in that order) it also holds between *q* and *p*.

Any contradiction necessarily implies any proposition.
Any necessary truth is necessarily implied by any proposition.

If (but not only if) certain truth-table patterns turn up, then relations 1, 2, 5, 6 and 9 occur. With possible-truth tables, the following method provides an iff test for all 9 relations. The first 7 relations are collectively exhaustive though not mutually exclusive. The last 2 relations are exhaustive and exclusive. In testing for the first 7 relations, adopt the order shown: this allows half-tests for contrariety and subcontrariety and a default detection of indifference (if 1 – 6 fail then 7 holds).

1.	*p* is equivalent to *q*	(matching main-columns)
2.	*p* is contradictory to *q*	(opposite main-columns)
3.	*p* is contrary to *q*	(no row with both true)
4.	*p* is subcontrary to *q*	(no row with both false)
5.	*p* implies *q*	(no row with *p* true and *q* false)
6.	*p* is implied by *q*	(no row with *q* true and *p* false)
7.	*p* is indifferent to *q*	
8.	*p* is consistent with *q*	(there is a row with both true)
9.	*p* is inconsistent with *q*	(no row with both true)

A *set* of *n* propositions is *indifferent* iff every truth-value permutation of the propositions is possible. A standard truth table for a proposition is a possible-truth table iff the matrix propositions are indifferent.

We often abbreviate "This is a tautology:" to "T:". Tautologyhood is preserved under uniform replacement of PVs with PL-wffs e.g., T: $p \vee \sim p$ implies T: $\alpha \vee \sim \alpha$ where α is any PL-wff.

Two PL-wffs α and β are *tautologically equivalent* iff T: $\alpha \equiv \beta$ i.e. iff α and β have matching main-columns in their truth table. Propositions are tautologically equivalent iff they have tautologically equivalent forms. For propositions, tautological equivalence implies necessary equivalence (but not vice versa).

We say that α *tautologically implies* β iff T: $\alpha \supset \beta$ i.e. iff there is no row with $\alpha = 1$ and $\beta = 0$. A proposition tautologically implies another iff it has a form which tautologically implies a form of the other. For propositions, tautological implication implies necessary implication (but not vice versa). For PC, the symbols $\Leftrightarrow$, $\Rightarrow$ and $\Leftarrow$ will be used to denote tautological equivalence, tautological implication and its converse respectively: these modal operators are given lower evaluation priority than our truth-functional operators.

The *converse, inverse* and *contrapositive* of $p \supset q$ are $q \supset p$, $\sim p \supset \sim q$ and $\sim q \supset \sim p$ respectively: of these, only the contrapositive is equivalent to the original.

Some important logical laws are listed below. Since a PL-sentence or proposition with a tautologous form is a tautology, these results have formal, sentential and propositional significance.

Name	*Abbreviation*	*Law*		
Law of Excluded Middle	LEM	T:		$p \vee \sim p$
Law of Non-Contradiction	LNC	T:		$\sim(p \,\&\, \sim p)$
Law of Bi-Valence	LBV	T:		$p \not\equiv \sim p$
Double Negation	DN	$\sim\sim p$	$\Leftrightarrow$	p
Commutativity (True for $* = \&, \vee, \equiv, \not\equiv$)	Com *	$p * q$	$\Leftrightarrow$	$q * p$
Associativity (True for $* = \&, \vee, \equiv, \not\equiv$)	Assoc *	$p * (q * r)$	$\Leftrightarrow$	$(p * q) * r$
De Morgan's Laws	DeM	$\sim(p \,\&\, q)$	$\Leftrightarrow$	$\sim p \vee \sim q$
		$\sim(p \vee q)$	$\Leftrightarrow$	$\sim p \,\&\, \sim q$
Contraposition	Contrap	$p \supset q$	$\Leftrightarrow$	$\sim q \supset \sim p$
Export-Import	Exim	$(p \,\&\, q) \supset r$	$\Leftrightarrow$	$p \supset (q \supset r)$

4

Using Tables To Assess Arguments

4.1 INTRODUCTION

Perhaps the most practical concern of logic is the analysis of arguments. In Chapter 1 we had an informal look at arguments, and noted that this text focusses its attention on deductive arguments (i.e. arguments where the conclusion is claimed to follow with certainty from the premises). In this chapter truth tables and possible-truth tables, introduced previously to classify propositions and relations, will be used to assess arguments.

When dealing with arguments there are two main things to be done in propositional logic: first, the argument is translated into PL; secondly, it is assessed for validity. A third logical move that can be made is to test for consistency of premises. We will look at things in this order, and then note some valid argument-forms of special importance.

At this point it would be a good idea for you to review § 1.7 and § 1.8 as well as the early part of § 2.6. In particular, you should feel comfortable with the terms "argument", "standard form", "argument-form", "valid", "invalid", "logical error", "factual error", "sound" and "unsound" as they are used in logic, and be aware that when synonymous translations are not possible, equivalent or implied translations are often used.

4.2 TRANSLATING ARGUMENTS

In translating an argument into PL we must first, at least mentally, put it into standard form. As discussed in § 1.7 this involves separating out the premises and conclusion. Do this now for argument (1). Remember to look for conclusion-markers like "hence" and "so", and premise-markers like "since" and "because".

> If he is a Christian he believes in one God. If he is a Hindu he believes in one God. As he is either a Christian or a Hindu it follows that he believes in one God. (1)

Did you spot "it follows that" as a conclusion-marker and "as" as a premise-marker? In standard form (1) is written as follows.

> If he is a Christian he believes in one God.
> If he is a Hindu he believes in one God.

$$\begin{array}{ll} & \text{He is either a Christian or a Hindu.} \\ \hline \therefore & \text{He believes in one God.} \end{array} \qquad (1a)$$

Before reading on, try putting this into PL, assuming that our PL-operators (e.g., ⊃) do justice to the English operators (e.g., if ... then).

You should have chosen a dictionary like the following.

C = He is a Christian
H = He is a Hindu
G = He believes in one God

With this dictionary, (1) translates into PL as:

$$\begin{array}{ll} & C \supset G \\ & H \supset G \\ & C \vee H \\ \hline \therefore & G \end{array} \qquad (1b)$$

Although a dictionary may sometimes be supplied, in other cases you will need to supply your own. As with any translation into PL, make sure you *set your dictionary out in full*. Each propositional constant must denote a whole proposition. For example it would be *incorrect* to write the following:

C = Christian
H = Hindu
G = believes in one God

On the right hand side of the "=" we must have a complete English sentence, not just a word or phrase.

A PL-translation where each propositional constant denotes an atomic proposition is called an *explicit PL-translation*. For example, (1b) is an explicit PL-translation of (1). In §3.5 we saw that if two or more atomic propositions occur only in a certain compound proposition then there is no need to specify these individually in translation. To illustrate this in relation to arguments consider the following example.

> If it is raining and the wind is blowing from the south then water will come under the door. Since it is raining and the wind is coming from the south it is clear that water will come under the door. (2)

An explicit PL-translation for (2) may be provided as follows.

R = It is raining
S = The wind is blowing from the south
D = Water will come under the door

$$\begin{array}{ll} & (R \mathbin{\&} S) \supset D \\ & R \mathbin{\&} S \\ \hline \therefore & D \end{array} \qquad (2a)$$

Since however, R and S occur only in the compound R & S, the following translation will be adequate.

A = It is raining and the wind is blowing from the south
D = Water will come under the door

$$\begin{array}{ll} & A \supset D \\ & A \\ \hline \therefore & D \end{array} \qquad (2b)$$

The validity of (2) is more easily demonstrated with (2b) than with (2a). If this is not obvious now, it will be after you have seen how to test validity with truth tables. Because less detailed translations can save work, the point bears repeating: *when two or more atomic propositions occur only in one compound they do not require separate dictionary entries.*

Sometimes a simple negation remains unbroken throughout an argument. Though we could translate this negation by a single letter, instead we usually include a $\sim$ so as to agree with our general preference to *choose affirmative propositions for the dictionary.* For example, argument (3) would normally be translated with $\sim T$ as shown.

Either the relay is working or the lights will not turn on.
Because the relay is not working, the lights will not turn on. (3)

R = The relay is working
T = The lights will turn on

$$\begin{array}{l} R \vee \sim T \\ \underline{\sim R \qquad} \\ \therefore \sim T \end{array} \tag{3a}$$

If you feel a bit rusty on translation, you might like to review the summary in §2.7 before proceeding.

EXERCISE 4.2

1. Translate each of the arguments in Exercise 1.7 Question 2 into PL, using the letters suggested.
2. Provide a shorter dictionary and adequate translation into PL for arguments 2(f) and 2(q) of Exercise 1.7.

4.3 TRUTH TABLES AND VALIDITY

In §1.8 we noted that an argument is valid iff the truth of the premises guarantees the truth of the conclusion. Having dealt with modal relations, we are now in a position to provide more precise definitions for validity.

Definition: An argument is *valid* iff the premises necessarily imply the conclusion.
An argument which is not valid is *invalid*.

The following equivalent definition for validity follows from our defnition of necessary implication in §3.7.

Definition: An argument is *valid* iff there is no possible world with all the premises true and the conclusion false.

This notion of validity as a particular case of implication may be set out as below, where propositions p_1, ..., p_n are the premises, proposition q is the conclusion, and the slash "/" performs in horizontal layout the job that "__________" does in vertical layout.

The argument $p_1, ..., p_n \; / \; \therefore \; q$ is valid iff
$(p_1$ &...& $p_n)$ necessarily implies q
i.e. there is no possible world where p_1 &...& p_n & $\sim q$

With respect to arguments, the term "*counterexample*" means "*a possible world with all the premises true and the conclusion false*". So an argument is valid iff it has no counterexample. To prove an argument valid we need to show that there is no counterexample. To prove an argument invalid we need to show that there is a counterexample.

To see how truth tables can help determine validity, let's begin with the following argument which was symbolized in §4.2.

$$\begin{array}{ll} & C \supset G \\ & H \supset G \\ & \underline{C \vee H} \\ \therefore & G \end{array} \qquad (1)$$

When constructing a truth table for an argument we use a common matrix, and list the conclusion last with a "∴" above it, thus:

C	G	H	$C \supset G$	$H \supset G$	$C \vee H$	∴ G
1	1	1	1	1	1	1
1	1	0	1	1	1	1
1	0	1	0	0	1	0
1	0	0	0	1	1	0
0	1	1	1	1	1	1
0	1	0	1	1	0	1
0	0	1	1	0	1	0
0	0	0	1	1	0	0

We have said that an argument is valid iff there is no counterexample. Since the rows of a truth table encompass all possible worlds, if there is a counterexample it must be represented by one of the rows. Thus, *if there is no row with premises all true and conclusion false the argument is valid.* If you look at the table above you will see that, although the premises are true on rows 1, 2 and 5, there is no row with premises true and conclusion false. So argument (1) must be valid.

Arguments like (1), which can be shown to be valid because of their PL-structure, will be called *PC-valid.* To be precise, an argument is PC-valid iff the conjunction of the premises *tautologically* implies the conclusion.

Now consider the following "crazy" argument.

Earth does have a moon or it doesn't.
So Earth does and does not have a moon. (2)

Symbolizing this as $M \vee \sim M \; / \; \therefore \; M \; \& \sim M$ we obtain the following table.

M	$M \vee \sim M$	∴ $M \; \& \sim M$
1	1	0
0	1	0
	↑	↑

Since there are some possible worlds, and the rows cover all possible worlds, at least one row must be possible. This is true of any truth table. But notice here that we have premise true and conclusion false on *all* rows. So at least one of the rows must provide a counterexample. Thus argument (2) is invalid. Arguments like (2), which have tautologous premises and a PC-contradiction as conclusion, can thus be proven invalid on account of their PL-structure: such arguments are called *PC-invalid.* They are almost never encountered in normal situations.

Let's look at another argument.

My favourite colour is green.
Therefore my favourite colour is not red. (3)

Symbolizing this as $G \ / \ \therefore \sim R$ we obtain the following truth table.

			$\therefore$
G	R	G	$\sim R$
1	1	1	0
1	0	1	1
0	1	0	0
0	0	0	1

On row 1 the premise is true and the conclusion is false. Does this row provide a counterexample? No! Your logical intuitions should tell you that the matrix on row 1 is impossible. So finding a row with premises true and conclusion false does not establish a counterexample unless we know that the row is possible. If there is at least one row which does not have premises true and conclusion false then this might be the only possible row. So in a standard truth table, if some rows have premises true and conclusion false, and some rows don't, it cannot be determined whether the argument is valid simply from this table. If this situation arises in an argument's ***explicit*** truth-table (i.e. the truth table for its explicit PL-translation), then the argument is said to be ***PC-indeterminate***. Some PC-indeterminate arguments will, like (3), be valid; others will be invalid; but PC will be unable to determine which, since it does not provide a means of peeking inside atomic propositions to decide which rows are possible.

For simplicity, let us agree to read the phrases "premises true" and "true premises" as "premises all true" unless otherwise qualified. As far as PC goes, the best that we can do is sort out arguments into three types as indicated below.

argument	*main-column values of truth table*
PC-valid	no row with premises true and conclusion false
PC-invalid	all rows with premises true and conclusion false
PC-indeterminate	*explicit* truth table has just some rows with premises true and conclusion false.

EXERCISE 4.3

1. Use truth tables to test the arguments translated in Exercise 4.2 Question 1 for validity. Classify each as PC-valid, PC-invalid or PC-indeterminate.
2. Given the argument $p_1 \ldots, p_n \ / \ \therefore q$, which of the following does NOT provide a necessary and sufficient condition for validity?

   A. $(p_1 \ \&\ldots\& \ p_n) \supset q$ is a necessary truth
   B. $\sim((p_1 \ \&\ldots\& \ p_n) \supset q)$ is a contradiction
   C. $p_1 \ \&\ldots\& \ p_n \ \& \sim q$ is a contradiction
   D. $(p_1 \ \&\ldots\& \ p_n)$ tautologically implies q

4.4 POSSIBLE-TRUTH TABLES AND VALIDITY

In the previous section we saw that standard truth-tables cannot determine the validity or invalidity of arguments that are PC-indeterminate. This deficiency can be overcome if, by using our logical intuitions or a calculus more powerful than PC, we can eliminate impossible rows to produce a possible-truth table. As we saw in §3.6, a row is present in a possible-truth table iff it represents some possible world. So once a possible-truth table for the argument has been constructed we may say that a counterexample exists iff there is a row with premises true and conclusion false. If no such row exists the argument is valid; if such a row does exist the argument is invalid. So possible-truth tables divide arguments up as follows:

argument	*main-column values of possible-truth table*
valid	no row with premises true and conclusion false
invalid	at least one row with premises true and conclusion false.

Let's see how this works on the following argument, which you may remember from the previous section.

My favourite colour is green.
Therefore my favourite colour is not red. (1)

This has the following possible-truth table:

	G	R	G	$\therefore$ $\sim R$
×	1	1	1	0
	1	0	1	1
	0	1	0	0
	0	0	0	1

Notice that row 1 has been eliminated because there is no possible world where my favourite colour is simultaneously green and red (we assume that "green" and "red" are used in a total rather than partial sense so that a mixture of green and red does not count as green or as red). Row 2 has my favourite colour as green, row 3 has it as red, and row 4 has it as neither (e.g., it could be violet): all these are possible. None of the three rows in our possible-truth table have the premise true and the conclusion false. So there is no counterexample: the argument is valid.

Now look at the following argument.

> It's not the case that both Snoopy and the Red Baron will be shot down. It is a fact that Snoopy will not be shot down. From this it may be inferred that the Red Baron will be shot down. (2)

Going on your intuitions, would you say that this is a valid argument? Let's see how good your intuitions were, by setting out a formal analysis of the argument.

Dictionary: S = Snoopy will be shot down.
R = The Red Baron will be shot down.

Translation:

$$\begin{array}{rl} & \sim(S \,\&\, R) \\ & \underline{\sim S} \\ \therefore & R \end{array}$$

Truth Table:

S	R	$\sim$	$(S \,\&\, R)$	$\sim S$	$\therefore$ R
1	1	0	1	0	1
1	0	1	0	0	0
0	1	1	0	1	1
0	0	1	0	1	0
		↑			

On the 4th row all the premises are true and the conclusion is false. Does this yield a counterexample? If you look across to the matrix you will find the following assignments for this row:

S	R
0	0

So this row represents those possible worlds (if any) where neither Snoopy nor the Red Baron get shot down. Intuitively, we can see that there are such possible worlds. Hence row 4 does indeed provide a counterexample, and the argument is invalid.

There are three things worth noting from this example. First, when an argument is found to be invalid we should always state a counterexample: this is best done by specifying the matrix assignments as above rather than a row number, since different matrix orders are sometimes used.

Secondly, regardless of how a counterexample is found, if it can be seen to be a possible way of having the premises true and the conclusion false this is sufficient to establish invalidity. This is particularly important to bear in mind when trying to show a non-logician that an argument is invalid: there would be little point in showing him a possible-truth table; rather, the counterexample found should be put into words for him to check out himself. In the above case for instance, having deduced the counterexample $S = 0$, $R = 0$, one should remind him of the possibility of neither Snoopy nor the Red Baron being shot down and then get him to see that in this case the premises are true but the conclusion is false. If we are on our toes we can often invent a counterexample simply by using our imagination; if our imagination is lacking we can try to construct a possible-truth table: once constructed, if there is a counterexample it will show it. In part, our work on possible-truth tables is designed to educate our imagination towards the production of counterexamples so that ultimately we can do without these tables.

The third point arising from the example is that *it is usually not necessary to check all the matrix rows for possibility.* Though in fact every row in the above example is a possible-row, we did not need this information to deduce invalidity. Let us use the term "*counter row*" to mean "a row which provides a counterexample". As soon as a counter row is found, there is no need to look at any other rows. So there is usually no need to ensure that the truth table is fully converted into a possible-truth table.

The following method for testing arguments for validity may now be stated. Since it can involve checking matrix rows for possibility it strictly goes beyond the scope of PC.

Method:

1. **Translate the argument into PL.**
2. **Draw a truth table for the premises and conclusion (using a common matrix).**
3. **The argument is *valid* iff there is no *possible*-row on which all the premises are true and the conclusion is false.**
4. **If at least one such *possible*-row exists the argument is *invalid* and the row provides a *counterexample.* A counterexample is formally stated by specifying the matrix assignments on a counter row; an argument may have more than one counterexample but it is only necessary to find one to establish invalidity.**

With this method, we begin with a standard truth table, and if there are any rows with true premises and false conclusion we check these one at a time for possibility until we find that one is possible and immediately declare the argument invalid, or that none are possible and declare the argument valid. If there were no rows with true premises and false conclusion the argument may be declared valid straight away.

Short Cuts

In §3.3 the following one-operand evaluation rules were introduced to speed up evaluation of formulae: $\ldots\ 0 \,\&\, \ldots = 0$; $\ldots \,\&\, 0 = 0$; $1 \vee \ldots = 1$; $\ldots \vee 1 = 1$; $0 \supset \ldots = 1$; $\ldots \supset 1 = 1$. In addition to these rules, the following *shortened tabular method* may be used to save work when assessing arguments. It is based on the idea that our tabular method is really a systematic search for a counterexample. To provide a counterexample a row must satisfy two conditions: it must have premises true and conclusion false; and it must be possible. Thus if a row has a false premise or a true conclusion or an impossible assignment of values to its matrix propositions, it cannot provide a counterexample and there is no need to do any further work on it. Let us place a "x" to the *right* of a row to indicate it is eliminated because of a false premise or a true conclusion. As usual, a "x" to the left of a row indicates elimination because the row is not possible. Since there is only one conclusion it is usually best to evaluate this first. The shortened tabular method may now be summarized as below:

1. Evaluate conclusion first, eliminating any row where conclusion = 1
2. Evaluate premises one at a time, eliminating any row where a premise = 0
3. Eliminate any remaining rows that are impossible, stopping as soon as a counterexample is found (argument is then invalid). If no rows remain, argument is valid.

Example: Test the following argument for validity.

If I have a blue car then I have a car.
If I have a red car then I have a car.
I do have a car.
So I have either a blue car or a red car.

Dictionary: B = I have a blue car
R = I have a red car
C = I have a car

Translation:

$$\begin{array}{l} B \supset C \\ R \supset C \\ C \\ \hline \therefore\ B \vee R \end{array}$$

Table:

B	R	C	$B \supset C$	$R \supset C$	C	$\therefore$ $B \vee R$	
1	1	1				1	x
1	1	0				1	x
1	0	1				1	x
1	0	0				1	x
0	1	1				1	x
0	1	0				1	x
0	0	1	1	1	1	0	
0	0	0			0	0	x

Rows 1 – 6 were eliminated because they have a true conclusion. This left rows 7 and 8. Row 8 was then eliminated because its third premise is false. This left row 7 which has the matrix assignment:–

B	R	C
0	0	1

This represents those possible-worlds (if any) where I have a car which is neither blue nor red: clearly there is such a possible-world e.g., I might have just a green car. So row 7 provides a counterexample and the argument is invalid. Notice that it was not necessary to test the other rows for possibility (Rows 2, 4 and 6 are in fact impossible).

Sometimes we may have a purported counterexample which needs to be tested. Perhaps we invented it from our imagination or somebody else suggested it. To check whether it is indeed a counterexample we first *substitute* it into the argument to see if it makes the premises true and the conclusion false. If it passes this test we then make a final check on its matrix assignment to see if it is possible.

Example: Using the dictionary of the previous example, check whether

B	R	C
0	0	1

is a counterexample to the argument:

$(B \vee R) \supset C$
$C \,\&\, {\sim}B$
$\therefore\ R$

Substituting in, we obtain

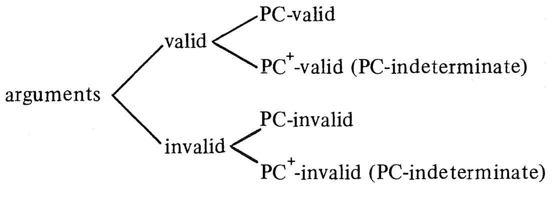
B	R	C	$(B \vee R) \supset C$	$C \,\&\, {\sim}B$	$\therefore R$
0	0	1	0 1	1 1	0
			↑	↑	

This makes the premises true and the conclusion false. Moreover, the row is possible. So we do have a counterexample and the argument is invalid.

Note: You are now equipped with a powerful technique for testing the validity of many arguments. Don't simply apply it blindly. Given an argument, read through it and try to "see" mentally whether it is valid or not. Then, after working through the formal solution on paper, check to see whether the result agrees with your mental solution. In this way the mind is sharpened and careless formal errors may be detected.

NOTES

PC-indeterminate arguments are valid or invalid for reasons that go beyond their PL structure: those that are valid are thus appropriately called PC^+-valid and those that are invalid are PC^+-invalid. This yields the following categorization:



Just as with determining validity of arguments, it is usually not necessary to check that every row is possible when determining modal properties of propositions or modal relations between propositions: all that need to be checked for possibility are those rows which, if possible, would provide a counterexample to the property or relation. For example, a proposition is a necessary truth iff there is no possible row on which the proposition is false: when testing for a necessary truth we need check for possibility only those rows on which the proposition is false.

EXERCISE 4.4

1. Given that A, B and C are *indifferent* propositions (and hence the possible-truth table is the same as the standard truth table), test the following arguments for validity. Where invalid, state a counterexample.

(a)
$$\begin{array}{ll} & A \supset B \\ & B \\ \hline \therefore & A \end{array}$$

(b)
$$\begin{array}{ll} & A \supset B \\ & A \\ \hline \therefore & B \end{array}$$

(c)
$$\begin{array}{ll} & A \supset B \\ & \sim B \\ \hline \therefore & \sim A \end{array}$$

(d)
$$\begin{array}{ll} & A \supset B \\ & \sim A \\ \hline \therefore & \sim B \end{array}$$

(e)
$$\begin{array}{ll} & A \supset B \\ & \sim(A \,\&\, B) \\ \hline \therefore & \sim(A \vee B) \end{array}$$

(f)
$$\begin{array}{ll} & A \vee B \\ & B \equiv \sim A \\ \hline \therefore & A \,\&\, \sim B \end{array}$$

(g)
$$\begin{array}{ll} & A \supset B \\ & B \supset C \\ \hline \therefore & A \supset C \end{array}$$

(h)
$$\begin{array}{l} A \supset (B \vee C) \\ A \equiv \sim B \\ \hline \sim A \end{array}$$

(i)
$$\begin{array}{ll} & A \vee B \\ & A \supset (C \,\&\, \sim B) \\ & B \equiv C \\ \hline \therefore & C \end{array}$$

(j)
$$\begin{array}{ll} & A \supset \sim B \\ & B \supset C \\ \hline \therefore & \sim C \end{array}$$

(k)
$$\begin{array}{ll} & A \vee (B \,\&\, C) \\ & C \equiv D \\ & \sim B \\ \hline \therefore & A \supset D \end{array}$$

2. Symbolize and test the following arguments for validity, using the suggested propositional constants. Where invalid, provide a counterexample.

(a) He is either tired or lazy. He is in fact tired. So he's not lazy. (T, L)

(b) He is either tired or lazy. But he's not lazy. So he's tired. (T, L)

(c) He's not both tired and lazy. Hence he's not tired and he's not lazy. (T, L)

(d) Either he's not tired or he's not lazy. Therefore it's not the case that he's either tired or lazy. (T, L)

(e) I will succeed if and only if I try. But I won't try unless I see a reason for trying. Hence I will succeed only if I see a reason for trying. (S, T, R)

(f) Hinck is a logician only if Malcolm is. Now Malcolm is not a logician unless both Rod and Hinck are. Since Rod is a logician however, we may deduce that at least one of Malcolm and Hinck are logicians. (H, M, R)

(g) I get annoyed with the class only if they're noisy. If they're not noisy then they learn a lot. Hence either I get annoyed with the class or they learn a lot. (A, N, L)

*(h) If I have at least seven marbles then I have eight marbles. Hence either I have at least seven marbles or I don't have eight marbles. (S, E)

3 Symbolize and test the following arguments. Provide a dictionary in each case, and if invalid state a counterexample.

(a) If I am very tired I will go to bed. However I don't get very tired unless I have been working hard. Now I have been working hard. So I will go to bed.

(b) Either Clark Kent is Superman or he's got an unusual attraction towards old phone booths. If he is Superman then he won't like the new transparent phone booths. He doesn't like the new transparent booths. We conclude therefore that he is Superman.

(c) Black Bart will visit the scene of the crime if he is the murderer, provided Sherlock Hemlock's theory is correct. Although he didn't visit the scene of the crime, nevertheless Sherlock's theory is still correct. Therefore Black Bart is not the murderer.

(d) If I travel to Alpha Centauri and back at relativistic speeds then, provided Einstein was right, I will be much younger than my twin when I return. So if I travel to Alpha Centauri and back but do not return much younger than my twin either I did not travel at relativistic speeds or Einstein was wrong.

(e) Students are worthy if and only if they work hard. Either students are hard working or they are neither worthy nor sensible. From all this you can see that students are sensible if they work hard.

(f) The Sausageites will attack the Bubbleonians only if they believe the Bubbleonians are inhuman invaders. Now unless the Sausageites are practical philosophers they will believe the propaganda of their government, and believing their government's propaganda is a sufficient condition for their believing that the Bubbleonians are inhuman invaders. Regrettably then, the Sausageites will attack the Bubbleonians, since the Sausageites are not practical philosophers.

*(g) If ghosts exist, they are spirits of dead people. Unless clothes have spirits however, ghosts, if they are spirits of dead people, will be naked. Obviously then, ghosts do not exist, since neither do clothes have spirits nor are ghosts naked.
(This argument was proposed by Wang Ch'ung, a Confucian philosopher of the first century A.D.)

*(h) If Lee writes the program he will write the program in a structured manner. If he does not write the program in a structured manner then, if he did write the program it would be difficult to follow. On the other hand, the program will not be difficult to follow if Lee writes it in a structured fashion. It may thus be inferred that, Lee writes the program in a structured way only if not only does he write the program but also it is not difficult to follow.

4. Given that A and B are indifferent, show by substitution which of the following specifications are counterexamples to the associated arguments.

(a)

A	B
0	1

$A \equiv \sim B, B \vee A \;/ \therefore\; B \supset A$

(b)

A	B
1	0

$\sim B \supset A, A \vee \sim B \;/ \therefore\; \sim(A \supset B)$

5. (a) Symbolize the following argument using the suggested letters.

If the figure is a rectangle it has four sides. If the figure is a square it has four sides. Now the figure does have four sides. So it must be either a rectangle or a square. (R, F, S)

(b) Now use your imagination to invent a counterexample, and then verify it by substitution.

*6. Using the dictionary of Question 5, symbolize the following argument, invent and specify as many counterexamples as you can, then check these by substitution. (Note: All squares are rectangles.)

> If the figure doesn't have four sides then it's not a square and not a rectangle. But it does have four sides. Hence it is both a square and a rectangle.

4.5 TWO CURIOUS CASES OF VALIDITY

Consider the following argument.

> It's raining.
> It's not raining.
> So Cygnus X-1 is a black hole. (1)

Assuming that the same time and place are involved for both the premises, the following dictionary will suffice:

R = It's raining
B = Cygnus X-1 is a black hole

So the argument may be symbolized as: $R, \sim R \ / \ \therefore B$. Before reading further, make an intuitive decision as to whether or not this argument is valid.

Let's check out your intuitions now with a table.

				$\therefore$
R	B	R	$\sim R$	B
1	1	1	0	1
1	0	1	0	0
0	1	0	1	1
0	0	0	1	0

Notice that there is no row with the premises true and the conclusion false. So the argument is valid! If you guessed correctly then give yourself a pat on the back. If you said the argument was invalid, you're forgiven: after all, what has the state of the weather got to do with whether (no pun intended) or not an X-ray source in the Cygnus constellation is a black hole? Clearly, something funny is going on, and if you look at the premises you will see what it is: the *premises are inconsistent*. There is no possible world where (at the same time and place) it is both raining and not raining. So there will be no possible-row in the table with the premises true (check this now): this automatically guarantees that there will be no possible-row with premises true and conclusion false, regardless of what the conclusion might be, and so the argument is valid.

From our earlier work (§ §1.6, 3.7), we may say that a set of propositions is inconsistent iff there is no possible world in which they are all true. Given any argument with an inconsistent set of premises then, there is no possible world with the premises true and the conclusion false (simply because there is no possible world with the premises true) and hence, by definition, the argument must be valid. So we have the following "paradox of validity":

> *any argument with inconsistent premises is valid.*

This is a particular case of the "paradox of necessary implication" mentioned in §3.7 that any contradiction necessarily implies any proposition (the premises are inconsistent iff their conjunction is a contradiction, and the argument is valid iff the conjunction of its premises necessarily implies the conclusion).

Since the above "paradox" is a natural consequence of our definition for validity you may feel that we should change this definition, seeing it counts arguments like (1) as valid. But saying that an argument is valid does not mean that it is free of errors. Recall from § 1.8 that whereas a sound argument (i.e. a valid argument with true premises) must have a true conclusion, no such guarantee can be made for unsound arguments, even if they are valid. If an argument has inconsistent premises they can't all be true and hence at least one is false (a factual error). Consequently *any argument with inconsistent premises is unsound.* Unsound arguments fail to establish their conclusion: those which, like (1), are unsound because of inconsistent premises, are said to commit the "fallacy of inconsistency".

The above "paradox of validity" thus ceases to surprise once we recognize validity as a weaker notion than soundness. On the positive side, there are good reasons for accepting this result into our logical system. Firstly, it serves as a useful warning against letting any inconsistency into our premises. This brings to mind the famous GIGO (Garbage In/Garbage Out) principle of computing i.e. if you input any "garbage" (errors) to the computer don't be surprised or blame the computer if you get garbage out. Similarly, if one begins with garbage in the form of inconsistent premises one may quite validly deduce anything (including garbage) from them. A second reason for allowing this "paradox", as well as another "paradox" to be discussed below, is that both are logically required if one accepts the following useful principle: a valid argument remains valid if further premises are added (a proof of this is referenced in the notes to this section).

To test for inconsistent premises we construct a truth table and then apply the following result:

> *the premises are inconsistent iff there is no possible-row with all the premises true*

Let's try this out now on the following, fairly tricky, argument.

> Paris is in France. If Pierre doesn't live in Paris then he lives somewhere in France. But he doesn't live anywhere in France. So Pierre lives in Paris. (2)

Before proceeding, you might like to make your own judgement as to the validity and soundness of this argument. Using the dictionary,

I = Paris is in France
P = Pierre lives in Paris
F = Pierre lives in France

we may symbolize (2) as: $I,\ \sim P \supset F,\ \sim F\ /\ \therefore\ P$. This leads to the following truth table:

I	P	F	I	$\sim P$	$\supset F$	$\sim F$	$\therefore P$
1	1	1	1	0	1	0	1
1	1	0	1	0	1	1	1
1	0	1	1	1	1	0	0
1	0	0	1	1	0	1	0
0	1	1	0	0	1	0	1
0	1	0	0	0	1	1	1
0	0	1	0	1	1	0	0
0	0	0	0	1	0	1	0
					↑		

First note that there is no row with premises true and conclusion false. So the argument is valid. Now look for a row with all the premises true. There is just one such row viz. row 2. Does this mean that the premises are consistent? No: not unless row 2 is possible. If you look across to the matrix you will find the assignment $I = 1, P = 1, F = 0$ which is impossible since this has Pierre living in Paris but not in France although Paris is in France. So there is no *possible*-row with all the premises true. Thus the premises are inconsistent and the argument, although valid, is unsound.

Now let's look at a different type of argument.

> My toenails are too long.
> Therefore it's raining or it's not raining. (3)

What do your intuitions tell you about the validity of this argument? Using an obvious dictionary we may symbolize it as: $T \ / \ \therefore R \vee \sim R$. This has the following truth table:

T	R	T	$\therefore$ $R \vee \sim R$
1	1	1	1
1	0	1	1
0	1	0	1
0	0	0	1
			↑

There is no row with the premises true and the conclusion false so, crazy as it may sound, (3) is a valid argument! What devious logical trick is going on here? This time the conclusion is the culprit. If you look at it you will see it is a necessary truth. So there is no possible-row with the conclusion false. Hence, regardless of the premises, there will be no possible-row with premises true and conclusion false, and the argument is automatically valid. This result may be generalized to give a second "paradox of validity":

any argument with a necessary conclusion is valid.

This is a particular case of the second "paradox of necessary implication" considered in §3.7 i.e. any necessary truth is necessarily implied by any proposition. Apart from the justification mentioned earlier, some sense can be made of this principle by regarding a necessary truth as something that is "true of its own nature" and hence as something that "follows from" nothing or anything. But probably the safest way to understand this "paradox" is to think of what terms like "valid" really mean in terms of our possible worlds framework. Thus even though (3) is valid (and sound if in fact my toenails are too long!) the argument is pointless because the conclusion is true in its own right: no premises are required to establish its truth. For obvious reasons, arguments like these are rarely, if ever, encountered in ordinary situations.

To test for this type of argument we construct a truth table and then apply the following result:

the conclusion is necessary iff there is no possible-row where it is false.

If the conclusion is a tautology, a standard truth table can detect it: if it is necessary but not PC-necessary, we will need to eliminate the impossible-rows where it is true.

It is not hard to show (the principal move is contraposition) that the two "paradoxes of validity" mentioned in this section are logically equivalent. It should also be realized that while anything follows from a contradiction, a contradiction does not follow from anything but a contradiction. In addition, while a necessary truth might be said to "follow from nothing", the only propositions that follow from necessary truths are themselves necessary truths.

NOTES

For a proof that the "paradoxes of validity" are equivalent to the rule that a valid argument remains valid if more premises are added, see Tapscott, B.L., *Elementary Applied Symbolic Logic* (Prentice-Hall, 1976) Appendix H.

EXERCISE 4.5

1. Given that A and B are indifferent propositions, use tables to determine which of the following arguments have inconsistent premises.
   (a) $A \not\equiv \sim A \;\; / \; \therefore A$
   (b) $\sim A \vee B, \;\; \sim(A \supset B) \;\; / \; \therefore B$
   (c) $A \supset \sim B, \;\; B \supset \sim A \;\; / \; \therefore A$
   (d) $B \supset A, \;\; B \supset \sim A, \;\; B \;\; / \; \therefore \sim B$

2. Symbolize the following arguments then use tables to determine their validity. Also state whether the premises are inconsistent and whether the conclusion is necessary.
   (a) Today is either Monday or not Monday. Therefore Mars has two moons.
   (b) Today is both Monday and not Monday. Therefore Mars has two moons.
   (c) Mars has two moons. Therefore today is either Monday or not Monday.
   (d) Mars has two moons. Therefore today is both Monday and not Monday.
   (e) It's pouring. If it's pouring then it's raining. If it's raining then it's wet. But it's not wet. So if it's raining it's not wet.
   (f) Cloning will be legalised only if its use is highly restricted. If the use of cloning is highly restricted it will not be legalised. Hence cloning will not be legalised.
   (g) Cloning will be legalised if and only if both restrictions are made on its use and science advances. Unfortunately, though science will advance, restrictions will not be made on the use of cloning. Cloning is bound to be legalised. Hence laws will be passed to allow cloning, without any restrictions on its use.
   *(h) 3 is greater than 7. Therefore 7 is greater than 3.
   *(i) Jigoro speaks Japanese but not Chinese. If Jigoro speaks Indian then he speaks Chinese. If he doesn't speak Indian he speaks no language at all. So he speaks either Indian or Chinese.

3. Use your answers to Question 2 to classify each of the arguments there (except for (f) whose premises are debatable) as sound or unsound. Make use of the facts that Mars does have two moons and that 3 is not greater than 7.

4. Which of the following are true?
   (a) A sound argument must have true premises.
   (b) An unsound argument must have inconsistent premises.
   (c) If the premises are inconsistent the argument must be valid but unsound.
   (d) If the conclusion is necessary the argument must be valid.
   (e) If the conclusion is necessary the argument must be unsound.
   (f) An unsound argument must have false premises.
   (g) Any conclusion may be validly deduced from a contradiction.
   (h) Some contingent propositions may be validly deduced from a necessary truth.
   (i) A necessary truth may be validly deduced from any proposition at all.
   (j) Contradictions may sometimes be validly deduced from a set of consistent propositions.

4.6 ARGUMENT-FORMS

In §2.5, PL-forms were discussed in relation to both PL-sentences and propositions. We now discuss PL-forms of whole arguments i.e. *PL-argument-forms.* First we will indicate what these are, classify them in terms of validity, and show how this classification links up to the evaluation of arguments. Then special mention will be made of some important valid argument-forms.

As early as §1.7 we met the notion of an argument-form as a common structure exhibited by different arguments. Arguments (1) and (2) of that section may now be symbolized in PL as follows:

$$\begin{array}{ll} & W \vee D \\ & \underline{\sim D} \\ \therefore & W \end{array} \qquad \begin{array}{ll} & C \vee S \\ & \underline{\sim S} \\ \therefore & C \end{array} \qquad (1), (2)$$

Both of these have the following form:

$$\begin{array}{ll} & p \vee q \\ & \underline{\sim q} \\ \therefore & p \end{array} \qquad (F1)$$

Now look at the following argument.

Logic is either interesting and relevant or boring.
Logic is not boring.
So logic is interesting and relevant.

Using an obvious dictionary we obtain the following translation.

$$\begin{array}{ll} & (I \,\&\, R) \vee B \\ & \underline{\sim B} \\ \therefore & (I \,\&\, R) \end{array} \qquad (3)$$

By treating $(I \,\&\, R)$ as a unit you can see that (3) also has (F1) as one of its forms. In general, an argument will be said to *have* (or be an *instance* of) a certain PL-form iff the argument can be expressed in PL by uniformly substituting the propositional variables in the form with PL-sentences. For example, (3) may be generated from (F1) by uniformly substituting "$(I \,\&\, R)$" and "B" for "p" and "q" respectively.

Notice that arguments (1), (2) and (3) also have the following form.

$$\begin{array}{ll} & p \\ & \underline{q} \\ \therefore & r \end{array} \qquad (F2)$$

Any argument with two premises will have this form. Clearly, this form does not fully detail the PL-structure of (1), (2) or (3). Maximum detail on an argument's PL-structure is provided by its *explicit* PL-form. This is obtained by uniformly substituting propositional variables (in the order $p, q, \ldots$) for the propositional constants in an explicit PL-translation of the argument: in this unique form each propositional variable relates to an *atomic* proposition (review §4.2 if needed). Looking back to arguments (1) and (2), since W, D, C and S are atomic it follows that (F1) is the explicit PL-form of both these arguments. However, the explicit PL-form of argument (3) will be as follows:

$$\begin{array}{ll} & (p \,\&\, q) \vee r \\ & \underline{\sim r} \\ \therefore & (p \,\&\, q) \end{array} \qquad (F3)$$

Truth tables may be constructed for argument-forms in the same way as for arguments. A PL-argument-form is said to be *valid* iff there is no row in its truth table where each premise-form = 1 and the conclusion-form = 0; otherwise it is *invalid.* For example, it may be seen from the table below that (F1) is a valid argument-form.

p	q	$p \vee q$	$\sim q$	$\therefore$ p
1	1	1	0	1
1	0	1	1	1
0	1	1	0	1
0	0	0	1	0

It should be obvious that, no matter what propositions are substituted for p and q in the above table the entries will be the same, and since there is no counter row the argument will be valid. This result may be generalized as follows:

any argument which has a valid PL-form is valid.

Since arguments (1), (2) and (3) each have the valid form (F1) they must all be valid. For simplicity, let us agree that in relation to argument-forms, the terms "premise" and "conclusion" may be used instead of "premise-form" and "conclusion-form".

Now look at the following argument-form.

$$\begin{array}{rl} & p \vee \sim p \\ \hline \therefore & p \mathbin{\&} \sim p \end{array} \qquad \text{(F4)}$$

As the table below indicates, this has each premise = 1 and conclusion = 0 on *all* rows of its table: such a form is said to be *contravalid.*

p	$p \vee \sim p$	$\therefore$ $p \mathbin{\&} \sim p$
1	1	0
0	1	0
	↑	↑

No matter what proposition is substituted for p, at least one of these rows will be possible and hence provide a counterexample. So any argument with this form will be invalid (in fact, PC-invalid). This result may be generalized as follows:

any argument which has a contravalid PL-form is invalid.

Contravalid argument-forms are rarely met. Usually, invalid PL-argument-forms have only some rows where premises = 1 and conclusion = 0 e.g.,

$$\begin{array}{rl} & p \supset q \\ & q \\ \hline \therefore & p \end{array} \qquad \text{(F5)}$$

p	q	$p \supset q$	q	$\therefore$ p
1	1	1	1	1
1	0	0	0	1
0	1	1	1	0
0	0	1	0	0

Row 3 shows that the argument-form (F5) is invalid. Does this mean that all arguments of this form will be invalid? No! It all depends on whether row 3 turns out to be possible when p and q are replaced by the corresponding propositions in the argument. This is illustrated by the following two arguments.

If Smith is a woman then Smith is human.
Smith is human.
So Smith is a woman. (4)

If the number is even it is divisible by 2.
The number is divisible by 2.
So the number is even. (5)

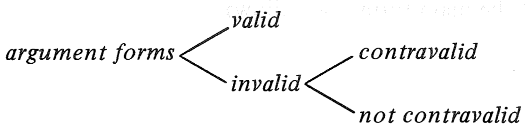
Using obvious dictionaries these may be translated as follows:

$$\begin{array}{rl} & W \supset H \\ & H \\ \hline \therefore & W \end{array} \qquad \begin{array}{rl} & E \supset D \\ & D \\ \hline \therefore & E \end{array}$$

Making the relevant substitutions in the table for (F5) we find that argument (4) is invalid because the row 3 assignment $W = 0$, $H = 1$ does provide a counterexample e.g., Smith could be a man. Argument (5) however turns out to be valid because the assignment $E = 0$, $D = 1$ is impossible. If a number is divisible by 2 it must be even: this logical truth enables the conclusion to be drawn from the second premise; the first premise is not required.

This type of result may be generalized as follows: ***any invalid PL-argument-form will, unless it is contravalid, have both invalid and valid instances.*** It is not hard to come up with further examples of valid arguments with invalid forms, particularly if we include forms that are not explicit. For instance, the valid arguments (1), (2) and (3) each have the invalid form (F2). It should be clear that the *explicit* PL-forms of PC-valid, PC-invalid, and PC-indeterminate arguments are valid, contravalid, and invalid (but not contravalid) respectively.

It may also be noted that our tabular definitions for validity states of PL-argument-forms are in agreement with the following general definitions which we adopt for all formal languages: an argument-form is valid iff every argument of that form is valid; an argument-form is contravalid iff every argument of that form is invalid; an argument-form is invalid but not contravalid iff some arguments of that form are invalid and some are valid.



Some valid argument-forms are so important that they are given special names. For example, each of the valid forms below is called *Denying a Disjunct* (DD).

$$\begin{array}{rl} & p \vee q \\ & {\sim}p \\ \hline \therefore & q \end{array} \qquad \begin{array}{rl} & p \vee q \\ & {\sim}q \\ \hline \therefore & p \end{array} \qquad \text{(DD)}$$

The right hand form has been met before as (F1). Remember that once an argument is recognized as having a valid form it may immediately be pronounced valid. So if we had earlier shown DD to be valid we could have assessed arguments (1), (2) and (3) as valid simply by recognizing that they had DD as a form: there is no need to produce a separate table for each. Thus having a list of commonly used valid argument-forms under our belt can save us work in much the same way that a library of precedents can enable lawyers to establish an immediate verdict. Fortunately, our list will be much shorter and simpler than a lawyer's. Valid argument-forms may also be used in reference to sub-arguments of more complicated arguments, and to justify lengthy deduction procedures: these matters will be taken up in Ch. 8 where a more comprehensive list of forms will be provided. Right now, it will be appropriate to consider just a few cases.

One set of valid forms can be extracted from our list of tautologies in §3.8 using the fact that if $\alpha \Leftrightarrow \beta$ then both $\alpha \;/\; \therefore\; \beta$ and $\beta \;/\; \therefore\; \alpha$ are valid. The name used for the tautology may also be used for the argument-form e.g.,

$$\begin{array}{lll} \sim\sim p \;/\; \therefore\; p & p \;/\; \therefore\; \sim\sim p & \text{(DN)} \\ p \,\&\, q \;/\; \therefore\; q \,\&\, p & & \text{(Com\&)} \\ p \supset q \;/\; \therefore\; \sim q \supset \sim p & \sim q \supset \sim p \;/\; \therefore\; p \supset q & \text{(Contrap)} \end{array}$$

Do you see why there is no need to specify $q \,\&\, p \;/\; \therefore\; p \,\&\, q$ as a second form of Com&?

Now look at the following four forms. Use your logical intuitions to determine which are valid.

$$\begin{array}{rlcrlcrlcrl} & p \supset q & \quad & & p \supset q & \quad & & p \supset q & \quad & & p \supset q \\ & \underline{p} & & & \underline{q} & & & \underline{\sim p} & & & \underline{\sim q} \\ \therefore & q & & \therefore & p & & \therefore & \sim q & & \therefore & \sim p \\ & \text{(AA)} & & & \text{(AC)} & & & \text{(DA)} & & & \text{(DC)} \end{array}$$

The labels under the forms are their abbreviated names: AA = Affirming the Antecedent; AC = Affirming the Consequent; DA = Denying the Antecedent; DC = Denying the Consequent. The names indicate what the second premise-form does to the antecedent or consequent in the first premise-form. The outer forms *Affirming the Antecedent* and *Denying the Consequent* are *valid*: they are often known by their Latin names *Modus Ponens* (MP) and *Modus Tollens* (MT) respectively. The inner two forms are invalid and are usually called *"the fallacy of affirming the consequent"* and *"the fallacy of denying the antecedent"*: it is fairly common for these to be mistakenly treated as valid because they closely resemble the other valid forms. Although AC and DA are invalid they are not contravalid, so rather special cases can be found of valid arguments with these forms e.g., argument (5) is valid though it has the form AC.

The last inference principle we discuss here is *Reductio Ad Absurdum* (RAA). There are many versions of RAA but the main form is as follows:

$$\begin{array}{rl} & p \supset \mathsf{F} \\ \hline \therefore & \sim p \end{array} \qquad \text{(RAA)}$$

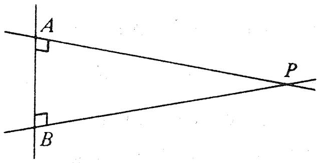
Here, and elsewhere, we use "F" to denote *any* necessary falsehood or contradiction. (A different style "F" is required if the dictionary already includes "*F*" for another proposition e.g., "Fred feels fine".) In essence RAA says that if p implies a contradiction p must be false. That this form is valid can be seen from the below table.

p	F	$p \supset \mathsf{F}$	$\therefore$ $\sim p$
1	0	0	0
0	0	1	1

Here we have eliminated the impossible rows where $\mathsf{F} = 1$.

RAA gives rise to the following proof technique. *If we start by making an assumption and then proceed by correct reasoning to an absurd result, we may conclude that our* *original assumption is wrong. So, in order to prove a result, assume the opposite and then show this leads to a contradiction.* This technique has many practical applications. In addition to underlying many of the logic methods to be discussed later (e.g., the method of assigning values, and the truth tree method), it is widely used in mathematics and science. Here are a couple of examples from mathematics and physics.

Example 1: Prove that if two coplanar lines are met by a third line in the same plane and at right angles to both, then these two lines are parallel.



Proof: Let the third line meet the others at points A and B.

Assume that the two lines are not parallel.

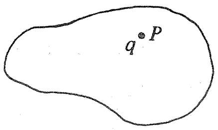
Then, since they are coplanar, they must meet at some point P forming the triangle APB. (from a previous theorem)

Since angle $A = 90^\circ$, angle $B = 90^\circ$ and angle P is greater than 0°, the angles of the triangle add up to more than 180°.

But this is impossible. (from a previous theorem)

Hence the two lines are parallel. (RAA)

Example 2: Prove that the electric field **E** is zero inside a statically charged conductor.



Proof: Assume that **E** is not zero at some point P inside.

Then the charge q at P will be pushed by the field and, being in a conductor, will be free to move.

But q must remain at rest. (*statically* charged conductor)

Hence the charge q is both moving and at rest.

This is impossible.

Hence **E** must be zero inside. (RAA)

NOTES

The argument-forms we have called "Denying a Disjunct" are often called "Disjunctive Syllogism" (DS). Sometimes only the first of these forms is given the label. The other is then derived by using Com ∨.

Following Patrick Suppes' work with logic in schools, we prefer the titles AA and DC to the less-easy-to-remember MP and MT. Nevertheless the latter terms are more standard. The names "*Modus Ponens*" and "*Modus Tollens*" derive from the Latin *modus* = manner, *ponere* = to affirm, *tollere* = to deny. The fuller names for MP and MT are "*Modus Ponendo Ponens*" and "*Modus Tollendo Tollens*". MP is sometimes called "Detachment".

Strictly, the symbol "F" represents an addition to PL. It may be treated as a PV whose range is restricted to contradictions i.e. it is a "contradiction variable".

EXERCISE 4.6

1. Use tables to classify each of the following argument-forms as valid or invalid. Where invalid, provide an invalidating matrix assignment.

(a) $p \supset \sim p \ / \ \therefore \sim p$
(b) $p \not\equiv q \ / \ \therefore p \,\&\, \sim q$
(c) $p \supset q, \sim q \ / \ \therefore \sim (p \vee q)$
(d) $p, q \equiv p \ / \ \therefore p \,\&\, q$
(e) $\sim (p \vee q) \vee (\sim p \supset q) \ / \ \therefore \sim (q \supset (p \vee q))$
(f) $\sim p \,\&\, \sim (q \,\&\, r), \sim q \supset p \ / \ \therefore \sim r$
(g) $(p \,\&\, r) \not\equiv q, \sim q \equiv r \ / \ \therefore \sim p$
(h) $(\sim\sim p \vee q) \equiv \sim r, \ r \supset \sim (q \vee r) \ / \ \therefore \sim q \supset p$

2. Which argument-form in Question 1 is contravalid?

3. Consider the following argument:

Apples are cheap. So bananas are either cheap or not cheap.

(a) Which of the following forms does this argument have?

(i) $p \ / \ \therefore q \vee \sim q$
(ii) $p \ / \ \therefore p$
(iii) $p \ / \ \therefore q$
(iv) $p \ / \ \therefore p \vee \sim p$

(b) Is the argument valid?

4. Consider the argument:

$$\begin{array}{l} \sim((A \equiv A) \,\&\, B) \\ \sim(A \equiv A) \\ \hline \therefore \sim B \end{array}$$

(a) Which of the following forms does the argument have?

(i) $p, q \ / \ \therefore r$
(ii) $\sim (p \,\&\, q), \ \sim p \ / \ \therefore \ \sim q$
(iii) $\sim ((p \equiv p) \,\&\, p), \ \sim (p \equiv p) \ / \ \therefore \ \sim p$
(iv) $\sim ((p \equiv p) \,\&\, q), \ \sim (p \equiv p) \ / \ \therefore \ \sim q$

(b) Classify those forms in (a) which were exhibited by the argument as valid or invalid.

(c) Is the argument valid?

5. Symbolize the following argument in PL:

It's raining.
So possibly it is raining.

(a) Is the explicit PL-form of this argument valid?
(b) Is this argument valid?

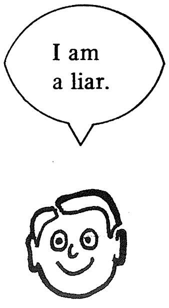
6. Answer TRUE or FALSE for each of the following.

(a) All arguments with a valid form are valid.
(b) All forms of a valid argument are valid.
(c) All arguments with an invalid form are invalid.
(d) All forms of an invalid argument are invalid.
(e) All arguments with a contravalid form are invalid.
(f) Some arguments are both invalid and contravalid.
(g) Any argument-form with a tautologous conclusion is valid.
(h) Any argument-form with a tautologous premise is valid.
(i) Any argument-form with a self-contradictory conclusion is valid.
(j) Any argument-form with a self-contradictory premise is valid.

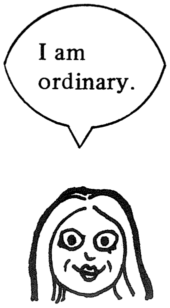
7. Establish the validity of the following arguments by naming a recognized valid argument-form exhibited by each.

   (a) $A \supset B, \ \sim B \ / \ \therefore \ \sim A$
   (b) $\sim\sim B \ / \ \therefore \ B$
   (c) $A \vee (B \,\&\, C), \ \sim A \ / \ \therefore \ B \,\&\, C$
   (d) $A \supset (B \,\&\, \sim B) \ / \ \therefore \ \sim A$
   (e) $\sim A \supset B \ / \ \therefore \ \sim B \supset \ \sim\sim A$
   (f) $A, A \supset B \ / \ \therefore \ B$
   (g) $\sim A \vee (B \equiv C) \ / \ \therefore \ (B \equiv C) \vee \ \sim A$
   (h) $\sim(A \,\&\, \sim B) \ / \ \therefore \ \sim A \vee \ \sim\sim B$
   (i) $\sim(B \not\equiv \sim A) \supset (B \vee C), \ \sim(B \not\equiv \sim A) \ / \ \therefore \ B \vee C$

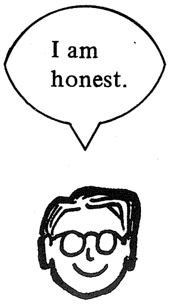
Puzzle 4. Of Alan, Betty and Cyril one is *honest* (always tells the truth), one is a *liar* (always lies), and one is *ordinary* (sometimes tells the truth and sometimes lies). Deduce who is what from the statements they make as shown below.



Alan



Betty



Cyril

4.7 SUMMARY

An argument's PL-translation is *explicit* iff each propositional constant denotes an atomic proposition. When two or more propositions occur only in one compound, they do not require separate dictionary entries. It is usually preferable to choose affirmative propositions for the dictionary.

An argument is *valid* iff the premises necessarily imply the conclusion i.e. there is no possible world with the premises true and conclusion false. If there is such a possible world it constitutes a counterexample and the argument is *invalid*.

Standard truth tables divide arguments up into three groups: *PC-valid* (no row with premises true and conclusion false); *PC-invalid* (all rows with premises true and conclusion false); *PC-indeterminate* (explicit truth table has just some rows with premises true and conclusion false). Some PC-indeterminate arguments are valid and some are invalid.

Possible-truth tables divide arguments up into two groups: *valid* (no row with premises true and conclusion false); *invalid* (at least one row with premises true and conclusion false).

It is not usually necessary to check all rows for possibility. The most efficient method is to begin with a standard truth table and then search for a *possible*-row with premises true and conclusion false: if such a row exists the argument is invalid; otherwise the argument is valid.

The *shortened tabular method*, in conjunction with the one-operand evaluation rules, provides a short cut search for a counterexample: evaluate conclusion first, eliminating any row where conclusion = 1, then evaluate premises one at a time, eliminating any row where a premise = 0; then eliminate any remaining rows that are impossible, stopping as soon as a counterexample is found (argument is then invalid); if no rows remain, argument is valid.

Two *"paradoxes of validity"* are: any argument with inconsistent premises is valid; any argument with a necessary conclusion is valid. Given a table for an argument, the premises are inconsistent iff there is no *possible*-row with them all true. If the premises are inconsistent, the argument, though valid, is unsound.

An argument has a certain PL-form iff it can be expressed in PL by uniformly substituting the propositional variables with PL-sentences. An argument-form is *valid* iff every argument of that form is valid; otherwise it is *invalid*. In rare cases an invalid argument-form will be *contravalid* i.e. every argument of that form is invalid. Usually invalid argument-forms have both invalid and valid instances. A PL-argument-form is repectively valid, contravalid, or invalid (but not contravalid) according as none, all, or just some of its truth table rows have each premise = 1 and the conclusion = 0. The explicit PL-forms of PC-valid, PC-invalid, and PC-indeterminate arguments are valid, contravalid, and simply invalid respectively.

Some important valid argument-forms may be extracted from our §3.9 summary using the fact that if $\alpha \Leftrightarrow \beta$ then both $\alpha \;/\; \therefore\; \beta$ and $\beta \;/\; \therefore\; \alpha$ are valid. For example, from $\sim\sim p \Leftrightarrow p$ we obtain $\sim\sim p \;/\; \therefore\; p$ and $p \;/\; \therefore\; \sim\sim p$ as valid forms. Other valid forms mentioned are listed below.

Name	*Abbreviation*	*Valid Argument-form*
Denying a Disjunct	DD	$p \vee q,\ \sim p \;/\; \therefore\; q$
	DD	$p \vee q,\ \sim q \;/\; \therefore\; p$
Affirming the Antecedent	AA	$p \supset q,\ p \;/\; \therefore\; q$
Denying the Consequent	DC	$p \supset q,\ \sim q \;/\; \therefore\; \sim p$
Reductio Ad Absurdum	RAA	$p \supset \mathsf{F} \;/\; \therefore\; \sim p$

We use "F" to denote *any* contradiction. AA and DC are often called "Modus Ponens" (MP) and "Modus Tollens" (MT). If an argument is seen to have a recognized valid form it may be immediately pronounced valid. RAA is often used in establishing a result by assuming its negation and then showing this leads to a contradiction.

Two invalid argument-forms of special note are $p \supset q, q \;/\; \therefore\; p$ (the fallacy of Affirming the Consequent) and $p \supset q,\ \sim p \;/\; \therefore\; \sim q$ (the fallacy of Denying the Antecedent).

5

The Method Of Assigning Values

5.1 INTRODUCTION

The "Method of Assigning Values" (MAV) has application in many fields, but in propositional logic it performs essentially the same jobs as the tabular methods discussed in the previous two chapters. We study MAV here because it often (though not always) performs these jobs *faster* than the tabular methods, especially when several propositional variables or constants are involved. Though related to the tabular methods, its *reductio ad absurdum* approach can yield very efficient solutions indeed, particularly when a written explanation of the solution is not required. For this reason it is highly favoured by experienced logicians for "mental" and "jot-down" solutions.

Its most useful role in propositional logic lies in testing for a *specific property or relation* (e.g., tautologyhood or equivalence) and testing for *validity.* After looking at the general rules for MAV we will consider specific applications separately: first the testing of modal properties of propositions; second, the testing of modal relations between two propositions; and finally the testing of arguments. Concurrently, the testing of various properties of and relations between PL-forms will also be mentioned.

5.2 GENERAL RULES

In general, we will start with a formula, assign a truth value to the main operator, and then deduce what values must be assigned to the rest of the formula on the assumption that our first assignment is correct. The object of the game is to determine whether or not the original assignment leads to a contradiction. We indicate the order of the steps taken by providing a row of numbers under our row of truth values; the original assignment is numbered 1.

Example: $p \supset q$

$\;1\;\;0\;\;0$ ← Truth Values

$\;2\;\;1\;\;2$ ← Order of steps.

Here we began by assigning 0 to the expression and inferred the values in step 2 from the fact that a material implication is false iff the antecedent is true and the consequent is false. Letting α and β denote any wffs of PL (either PL-sentences or PL-forms), this example may be generalized to the following rule:

$$\begin{array}{ccc}\alpha & \supset & \beta\\ & 0 & \end{array} \quad\rightarrow\quad \begin{array}{ccc}\alpha & \supset & \beta\\ 1 & 0 & 0\end{array}$$

In this manner the following list of *"one operator assignment rules"* may be drawn up to indicate what further assignments immediately follow from the assignment of a value to a single operator. Use the operator definitions to derive these rules for yourself.

Assignment Rules:

$$\begin{array}{cc}\sim & \alpha\\ 1 & \end{array} \quad\rightarrow\quad \begin{array}{cc}\sim & \alpha\\ 1 & 0\end{array}$$

$$\begin{array}{cc}\sim & \alpha\\ 0 & \end{array} \quad\rightarrow\quad \begin{array}{cc}\sim & \alpha\\ 0 & 1\end{array}$$

$$\begin{array}{ccc}\alpha & \& & \beta\\ & 1 & \end{array} \quad\rightarrow\quad \begin{array}{ccc}\alpha & \& & \beta\\ 1 & 1 & 1\end{array}$$

$$\begin{array}{ccc}\alpha & \& & \beta\\ & 0 & \end{array} \quad\rightarrow\quad \begin{array}{ccc}\alpha & \& & \beta\\ 0 & 0 & \end{array} \quad\text{or}\quad \begin{array}{ccc}\alpha & \& & \beta\\ & 0 & 0\end{array}$$

$$\begin{array}{ccc}\alpha & \vee & \beta\\ & 1 & \end{array} \quad\rightarrow\quad \begin{array}{ccc}\alpha & \vee & \beta\\ 1 & 1 & \end{array} \quad\text{or}\quad \begin{array}{ccc}\alpha & \vee & \beta\\ & 1 & 1\end{array}$$

$$\begin{array}{ccc}\alpha & \vee & \beta\\ & 0 & \end{array} \quad\rightarrow\quad \begin{array}{ccc}\alpha & \vee & \beta\\ 0 & 0 & 0\end{array}$$

$$\begin{array}{ccc}\alpha & \supset & \beta\\ & 1 & \end{array} \quad\rightarrow\quad \begin{array}{ccc}\alpha & \supset & \beta\\ 0 & 1 & \end{array} \quad\text{or}\quad \begin{array}{ccc}\alpha & \supset & \beta\\ & 1 & 1\end{array}$$

$$\begin{array}{ccc}\alpha & \supset & \beta\\ & 0 & \end{array} \quad\rightarrow\quad \begin{array}{ccc}\alpha & \supset & \beta\\ 1 & 0 & 0\end{array}$$

$$\begin{array}{ccc}\alpha & \equiv & \beta\\ & 1 & \end{array} \quad\rightarrow\quad \begin{array}{ccc}\alpha & \equiv & \beta\\ 1 & 1 & 1\end{array} \quad\text{or}\quad \begin{array}{ccc}\hat{\alpha} & \equiv & \beta\\ 0 & 1 & 0\end{array}$$

$$\begin{array}{ccc}\alpha & \equiv & \beta\\ & 0 & \end{array} \quad\rightarrow\quad \begin{array}{ccc}\alpha & \equiv & \beta\\ 1 & 0 & 0\end{array} \quad\text{or}\quad \begin{array}{ccc}\alpha & \equiv & \beta\\ 0 & 0 & 1\end{array}$$

$$\begin{array}{ccc}\alpha & \not\equiv & \beta\\ & 1 & \end{array} \quad\rightarrow\quad \begin{array}{ccc}\alpha & \not\equiv & \beta\\ 1 & 1 & 0\end{array} \quad\text{or}\quad \begin{array}{ccc}\alpha & \not\equiv & \beta\\ 0 & 1 & 1\end{array}$$

$$\begin{array}{ccc}\alpha & \not\equiv & \beta\\ & 0 & \end{array} \quad\rightarrow\quad \begin{array}{ccc}\alpha & \not\equiv & \beta\\ 1 & 0 & 1\end{array} \quad\text{or}\quad \begin{array}{ccc}\alpha & \not\equiv & \beta\\ 0 & 0 & 0\end{array}$$

Perhaps the most difficult of the above rules to derive is the one beginning with $\alpha \supset \beta = 1$ and leading to the alternatives $\alpha = 0$ or $\beta = 1$. One way to show this is as follows: $\alpha \supset \beta = 1$ iff $\sim(\alpha \supset \beta = 0)$ i.e. iff $\sim(\alpha{=}1 \;\&\; \beta{=}0)$ i.e. iff $\alpha{=}0 \vee \beta{=}1$. Another way is to verify the equivalence $p \supset q \Leftrightarrow \sim p \vee q$ by means of a truth table.

Notice that most of the above rules generate two alternatives: this is called *"splitting"*. When using a rule that produces two alternatives the previous work is copied down again; the original case may be used for the first alternative and the copy for the second alternative. To emphasize that the alternatives are separate cases a line may be drawn between them. Splitting occurs in step 2 of the following example where the fourth assignment rule on our list is used.

$$\begin{array}{ccccc}
p & \& & (q & \vee & r) \\
0 & 0 & & & \\
2 & 1 & & & \\
\hline
 & 0 & & 0 & \\
 & 1 & & 2 &
\end{array}$$

As you can imagine, if much splitting occurs MAV can become quite arduous. To save work then, when given a choice between using a rule that splits and one that doesn't split we choose the latter.

Efficiency Rule: Don't split until you have to.

The fact that all occurrences of the same propositional letter in a formula must be given the same truth value leads to the following rule.

Copy Rule: Once a value has been assigned to a propositional letter in an alternative, this value may be copied underneath all other occurrences of that letter in the alternative.

To indicate a copy we will *not* specify a step number. Instead we will simply *underline the copied value* e.g.,

$$\begin{array}{ccccc}
p & \supset & (p & \equiv & q) \\
1 & 0 & \underline{1} & 0 & \\
2 & 1 & & 2 &
\end{array}$$

The next rule, while not essential to MAV, is very useful. Intelligent application of this rule frequently leads to a substantial reduction in the amount of work (especially when it eliminates the need to split).

Resolution Rule: Where possible, a value may be assigned to a symbol by resolving it with respect to values already assigned in the same alternative.

For instance, in the previous example we have $(p \equiv q) = 0$ and $p = 1$. The only way for this to happen is to have $q = 0$. So we may assign 0 to q on this basis (see step 3 below):

$$\begin{array}{ccccc}
p & \supset & (p & \equiv & q) \\
1 & 0 & \underline{1} & 0 & 0 \\
2 & 1 & & 2 & 3
\end{array}$$

A list of such resolutions is given below (the resolved value is indicated with an asterisk). Don't bother to learn these rules off. Just check them through for yourself to be sure you could make the appropriate resolution when the particular value combination arose.

$$\begin{array}{cc}
\sim & \alpha \\
1 & 0 \\
* &
\end{array}
\qquad
\begin{array}{cc}
\sim & \alpha \\
0 & 1 \\
* &
\end{array}$$

$$\begin{array}{ccc}
\alpha & \& & \beta \\
1 & 0 & 0 \\
 & & *
\end{array}
\qquad
\begin{array}{ccc}
\alpha & \& & \beta \\
0 & 0 & 1 \\
* & &
\end{array}$$

$$\begin{array}{ccc}
\alpha & \vee & \beta \\
0 & 1 & 1 \\
 & & *
\end{array}
\qquad
\begin{array}{ccc}
\alpha & \vee & \beta \\
1 & 1 & 0 \\
* & &
\end{array}$$

$$\begin{array}{ccc} \alpha & \supset & \beta \\ 1 & 1 & 1 \\ & & * \end{array} \qquad \begin{array}{ccc} \alpha & \supset & \beta \\ 0 & 1 & 0 \\ * & & \end{array}$$

$$\begin{array}{ccc} \alpha & \equiv & \beta \\ 1 & 1 & 1 \\ & & * \end{array} \quad \begin{array}{ccc} \alpha & \equiv & \beta \\ 1 & 1 & 1 \\ * & & \end{array} \quad \begin{array}{ccc} \alpha & \equiv & \beta \\ 0 & 1 & 0 \\ & & * \end{array} \quad \begin{array}{ccc} \alpha & \equiv & \beta \\ 0 & 1 & 0 \\ * & & \end{array} \quad \begin{array}{ccc} \alpha & \equiv & \beta \\ 1 & 0 & 0 \\ & & * \end{array} \quad \begin{array}{ccc} \alpha & \equiv & \beta \\ 0 & 0 & 1 \\ * & & \end{array} \quad \begin{array}{ccc} \alpha & \equiv & \beta \\ 0 & 0 & 1 \\ & & * \end{array} \quad \begin{array}{ccc} \alpha & \equiv & \beta \\ 1 & 0 & 0 \\ * & & \end{array}$$

$$\begin{array}{ccc} \alpha & \not\equiv & \beta \\ 1 & 1 & 0 \\ & & * \end{array} \quad \begin{array}{ccc} \alpha & \not\equiv & \beta \\ 0 & 1 & 1 \\ * & & \end{array} \quad \begin{array}{ccc} \alpha & \not\equiv & \beta \\ 0 & 1 & 1 \\ & & * \end{array} \quad \begin{array}{ccc} \alpha & \not\equiv & \beta \\ 1 & 1 & 0 \\ * & & \end{array} \quad \begin{array}{ccc} \alpha & \not\equiv & \beta \\ 1 & 0 & 1 \\ & & * \end{array} \quad \begin{array}{ccc} \alpha & \not\equiv & \beta \\ 1 & 0 & 1 \\ * & & \end{array} \quad \begin{array}{ccc} \alpha & \not\equiv & \beta \\ 0 & 0 & 0 \\ & & * \end{array} \quad \begin{array}{ccc} \alpha & \not\equiv & \beta \\ 0 & 0 & 0 \\ * & & \end{array}$$

In addition to this list it is sometimes useful to assign values to an operator from the value(s) of its operand(s) e.g., $\begin{array}{ccc} \alpha & \equiv & \beta, \\ 1 & 1 & 1 \\ & * & \end{array}$ $\begin{array}{ccc} \alpha & \vee & \beta. \\ 1 & 1 & \\ & * & \end{array}$ We make no attempt to provide a list of such resolutions here since they have been covered earlier in our operator definitions, and our one-operand evaluation rules (§3.3).

When an alternative is found to generate a contradiction we cease work on it as it cannot provide a possible way of satisfying the original assignment: the alternative is then said to be *"closed"*. When all alternatives close this is called *"full closure"*. To indicate a contradiction we join the inconsistent assignments and place a cross underneath. From the point of view of PC, there are two different ways in which a contradiction may arise. The simplest case is when both 1 and 0 are assigned to the same symbol or subformula i.e.

$$\begin{array}{ccccc} \ldots & \alpha & \ldots & \alpha & \ldots \\ & \underbrace{1 \quad\quad\quad 0}_{\times} & & & \end{array} \qquad \text{or} \qquad \begin{array}{ccccc} \ldots & \alpha & \ldots & \alpha & \ldots \\ & \underbrace{0 \quad\quad\quad 1}_{\times} & & & \end{array}$$

Secondly, a contradiction may arise owing to an inconsistency between the value assigned to an operator and the values assigned to its operands e.g.,

$$\begin{array}{c} \alpha \;\supset\; \beta \\ \underbrace{1 \;\; 0 \;\; 1}_{\times} \end{array}$$

These different ways of generating a contradiction are illustrated in the two examples below, each of which shows that assigning 0 to the form $p \vee \sim p$ leads to a contradiction.

$$\begin{array}{cccc} p & \vee & \sim & p \\ 0 & 0 & 0 & \underline{0} \\ 2 & 1 & \multicolumn{2}{c}{\underbrace{2 \;\; \overline{}}_{\times}} \end{array} \qquad \begin{array}{cccc} p & \vee & \sim & p \\ 0 & 0 & 0 & 1 \\ \multicolumn{4}{c}{\underbrace{2 \;\; 1 \;\; 2 \;\; 3}_{\times}} \end{array}$$

Note that although the final result was the same (i.e. a contradiction was generated), there was more than one correct way of deriving this result. Because there is usually a choice of rules, MAV often permits several correct solutions to the one problem.

Closure Rule: Close an alternative as soon as (i) both 1 and 0 are assigned to different occurrences of the same expression, or (ii) an operator is assigned a value inconsistent with the value(s) assigned to its operand(s).

Within the context of MAV let us agree to use the words "case" and "alternative" synonymously: a single case which does not split may thus be regarded as a single alternative. Until a case is closed by producing a contradiction it is said to be *"open"*.

Completion Rule: Keep going until (i) there is at least one open case with all relevant values assigned,
or (ii) all cases close.

Note the implication for cases that split. The first alternative should be worked through completely: if this does *not* yield a contradiction, stop work as condition (i) is satisfied; if it does generate a contradiction you must test the other alternative because it might not close.

Having learnt the ground rules of MAV you are now in a position to understand its use for testing various properties and relations. The tests depend on the fact that the original assignment is a contradiction iff it generates a contradiction in all cases.

5.3 TESTING PROPOSITIONS

Before using MAV to test propositions let's look at PL-forms. Recall that a tautologous PL-form = 1 for all assignments of values to its PVs i.e. it's impossible for a tautology to have the value 0. So if we assign 0 to a tautology this must generate a contradiction in all cases. On the other hand, a self-contradictory PL-form = 0 for all assignments to its PVs: so assigning 1 to it must generate a contradiction in all cases. So MAV may be used to test PL-forms as follows.

PL-form	MAV test
tautology	assigning 0 to the main operator → full closure
contradiction	assigning 1 to the main operator → full closure
contingency	each of the above tests fail to give full closure

Example 1:

$$\begin{array}{ccccc} p & \supset & (q & \supset & p) \\ 1 & 0 & 1 & 0 & 0 \\ \end{array}$$
$$\underbrace{2\ \ 1\ \ 3\ \ 2\ \ 3}_{\times}$$

It's impossible for the formula to be 0.
∴ it is a tautology.

Example 2:

$$\begin{array}{cccc} p & \& & \sim & p \\ 1 & 1 & 1 & 0 \\ \end{array}$$
$$\underbrace{2\ \ 1\ \ 2\ \ 3}_{\times}$$

It's impossible for the formula to be 1.
∴ it is a contradiction.

Example 3:

p	$\supset$	$(p$	$\&$	$q)$
1	0	$\underline{1}$	0	0
2	1	‾	2	3

The formula can be 0.
So it is not a tautology.

p	$\supset$	$(p$	$\&$	$q)$
0	1	$\underline{0}$		
2	1	‾		
	1		1	
	1		2	

Obviously the first alternative will not close (regardless of what value q is given).
So the formula can be 1.
It is not a contradiction.

Hence the formula is contingent.

Notice how these tests involve RAA. For instance, in order to prove a formula *is* a tautology we assume it is *not* (by assigning 0 to it) and then show this generates a contradiction. Notice also that two tests are required to establish contingency: for this reason MAV is most efficient when we simply want to check that the formula is a tautology or check that it is a contradiction. Actually, there is a way of making one test sufficient for detecting all three types of form but we have no space to discuss it here (see Notes to § 6.3).

Now let's look at classifying propositions. Before applying MAV a translation into PL is required. From our work in §3.5 on such classifications, and the above work on PL-forms we can now state the following tests.

Proposition	MAV test on explicit PL-translation
tautology	assigning 0 to the main operator → full closure
PC-contradiction	assigning 1 to the main operator → full closure
PC-indeterminacy	each of the above tests fails to give full closure

Example 4: Use MAV to check whether the following proposition is a contradiction.

His aura is neither green nor blue, but it is green.

G = His aura is green
B = His aura is blue

$\sim$	$(G$	$\vee$	$B)$	$\&$	G
1	0	0	0	1	1
2	4	3	4	1	2
			x		

It's impossible for the proposition to be true.
So it is a contradiction.

As we know, PC-indeterminacies may be necessarily true, necessarily false or contingent, but so long as we stay within PC we are unable to determine which. However, if we look inside the atomic propositions involved and are able to decide what assignments to them are collectively possible (i.e. consistent), then such determinations can be made. In this way truth tables were extended to possible-truth tables. Similarly, MAV can be augmented to yield the *Method of Assigning Possible-Values* (MAPV). All we need do is add the following rule.

Subatomic Closure Rule: Close any alternative which has an inconsistent assignment of values to its propositional constants due to the internal nature of the dictionary propositions.

With MAV there are only two ways of closing an alternative. The above rule provides a third way by allowing closure if an internal analysis of the dictionary propositions reveals that the assignments to them are collectively impossible. As with possible-truth tables, the success of this method depends on how good our abilities are for spotting such inconsistencies. If the Subatomic Closure Rule is obeyed then any open case arrived at by the Completion Rule will be a *possible* case i.e. it will represent some possible world. MAPV thus sorts propositions into the same categories as possible-truth tables do.

Proposition	MAPV test on any PL-translation
Necessary Truth	assigning 0 to the main operator → full closure
Contradiction	assigning 1 to the main operator → full closure
Contingency	each of the above tests fails to give full closure

For obvious reasons MAPV, like possible-truth tables, has no application to PL-forms.

Example 5: Use MAPV to check whether the following proposition is necessarily true.

If Pat smokes cigars then he either smokes or drinks.

C = Pat smokes cigars
S = Pat smokes
D = Pat drinks

$$\begin{array}{ccccc} C & \supset & (S & \vee & D) \\ 1 & 0 & 0 & 0 & 0 \\ \underline{2} & 1 & \underline{3} & 2 & 3 \\ \multicolumn{3}{c}{\underbrace{\hspace{4em}}_{\times}} & & \end{array}$$

Closure results from the fact that there is no possible world in which $C = 1$ and $S = 0$.

The proposition is a necessary truth.

Although we provide a full setting out in our examples and answers, this is primarily for your benefit so you can more easily follow our working. In practice, if you are performing MAV or MAPV for yourself alone (i.e. if you do *not* intend your solution to be seen by others) then there really is no need to provide the step numbers. Omitting the step numbers saves a lot of writing, particularly in cases that split. However even in such cases it is still advisable to underline copied values; also, the initial assignment to the main operator is best done in a different colour.

In the following exercise and later exercises in this chapter we have generally avoided setting questions which involve much splitting, since in such cases MAV solutions can be quite tedious: such questions are best tackled by shortened truth tables or by the truth tree method to be discussed in the next chapter. After a bit of practice you will usually be able to quickly decide whether MAV will give you an efficient solution. In particular, if the formula has a few occurrences of $\equiv$ or $\not\equiv$ then you would normally be best off using a method other than MAV since these operators split on either assignment.

EXERCISE 5.3

Note: Splitting is required only for those questions marked with an asterisk.

1. Use MAV to determine which of the following are tautologies.
   - (a) $p \supset p$
   - (b) $p \supset q$
   - (c) $\sim(p \;\&\; \sim p)$
   - (d) $\sim p \supset (p \supset q)$
   - (e) $(p \vee q) \supset p$
   - (f) $p \supset (p \vee q)$
   - (g) $(p \;\&\; q) \vee (\sim p \vee \sim q)$
   - (h) $p \supset (q \supset (r \supset (s \supset t)))$
   - *(i) $p \supset (q \vee r)\ .\not\equiv.\ r \supset p$
   - *(j) $p \supset q\ .\equiv.\ \sim(p \;\&\; \sim q)$
2. Use MAV to determine which of the following are contradictions.
   - (a) $\sim(p \supset p)$
   - (b) $\sim(p \supset \sim p)$
   - (c) $\sim p \;\&\; \sim(p \supset q)$
   - (d) $\sim(p \vee (q \equiv p))$
   - (e) $[p \supset (q \supset r)] \;\&\; \sim[(p \;\&\; q) \supset r]$
   - *(f) $p \supset \sim p$
   - *(g) $(q \supset p) \;\&\; \sim(p \;\&\; q)$
3. Use your intuitions to classify the following formulae, then check with an MAV solution.
   - (a) $(p \;\&\; q) \supset (q \vee r)$
   - *(b) $(\sim p \;\&\; q) \vee (p \supset q)$
   - (c) $\sim[q \vee (\sim r \vee s)] \;\&\; (r \supset q)$
4. Symbolize the following propositions using the suggested letters, then use MAV to classify them as a tautology, PC-contradiction or PC-indeterminacy. Use your own logical intuitions to decide on what value to assign first.
   - (a) Although exactly one of Anderson and Belnap is a logician it is true to say that not only Belnap but also Anderson are logicians. (*A*, *B*)
   - (b) If Anderson is not a logician then it's not the case that both Anderson and Belnap are logicians. (*A*, *B*)
   - *(c) Either both Anderson and Belnap are logicians or Anderson is not a logician. (*A*, *B*)
5. Given that *A* and *B* as defined for the previous question are indifferent, what can you say about any PC-indeterminacy detected there?
6. Each of the following propositions is either a necessary truth or a contradiction. Use your intuitions to decide which, and then symbolize (using the suggested letters) and use MAPV to check your answers.
   - (a) If I am happy then someone is happy, unless you are not happy. (*I*, *S*, *Y*)
     (Note: You may assume I am a person!)
   - (b) Although I am very happy and you are happy, I am not happy. (*V*, *Y*, *I*)
   - (c) If Sue is a woman then it is not the case that she is both human and male.
     (*W*, *H*, *M*)
   - (d) If Smith is a woman, and Smith is not human if Smith is a monkey, then Smith is

not a monkey. (W, H, M)

*(e) Despite the fact that Sam is a man it must be admitted that he is human only if he is not male. (S, H, M)

5.4 TESTING RELATIONS

MAV and MAPV may be used to sort relations into the categories discussed for truth tables and possible-truth tables in §3.7. However, while the same table can be used to test several formulae for all the nine relations treated earlier, a separate MAV set-out is usually required to test each relation. This makes MAV inefficient for testing relations unless we are interested in just one relation. For this reason we will discuss only the two most important relations i.e. implication and equivalence. The specification of tests for the other relations is left as an exercise for the interested reader.

Since α *tautologically implies* β (i.e. $\alpha \Rightarrow \beta$) iff $\alpha \supset \beta$ is a tautology, to test whether $\alpha \Rightarrow \beta$ we simply apply the MAV test to determine whether $\alpha \supset \beta$ is a tautology i.e. assign 0 and see whether full closure results.

Example 1: Determine whether $\sim p \vee q$ tautologically implies $p \supset (q \vee r)$.

$$\begin{array}{cccccccccc} (\sim & p & \vee & q) & \supset & [p & \supset & (q & \vee & r)] \\ 0 & \underline{1} & 1 & \underline{0} & 0 & 1 & 0 & 0 & 0 & 0 \\ 5 & & 2 & & 1 & 3 & 2 & 4 & 3 & 4 \\ \lfloor & & \bot & \rfloor & & & & & & \\ & & \times & & & & & & & \end{array} \qquad \therefore \sim p \vee q \Rightarrow p \supset (q \vee r)$$

Similarly, to test whether α *necessarily implies* β we apply the MAPV test to determine whether $\alpha \supset \beta$ is a necessary truth. Remember that if α and β are propositions and $\alpha \Rightarrow \beta$ then α necessarily implies β, but the converse result does not hold in general.

There are several different procedures for testing whether α is *equivalent* to β. The direct method is based on the fact that α is tautologically/necessarily equivalent to β iff $\alpha \equiv \beta$ is a tautology/necessary truth: here we assign 0 to $\alpha \equiv \beta$ and see whether full closure results. Although direct, this method begins with a split and so can be tedious. A second method is based on the fact that α is equivalent to β iff α implies β and β implies α: here we use the implication test in both directions and look for full closure both times. But the method we favour is as follows:

> look at α and β and make an educated guess as to which value they will least frequently exhibit;
> assign this value to both and apply MAV (or MAPV);
> α and β are equivalent iff the open cases remaining match exactly.

You may think of this third method as finding the table rows on which the formulae have the assigned value; obviously the formulae are equivalent iff these rows match. Assignments for each formula should be treated separately: do *not* copy values from one formula to the other. Testing for equivalence is perhaps the most useful application of MAV: besides providing a quick check for equivalent translations, the third method just discussed allows "lightning fast" proofs of many important logical laws to be produced.

Example 2: A student translates the proposition

Apples are on the menu unless bananas are.

as $\sim A \supset B$ but the prepared solution is $A \vee B$. Is his translation correct? His translation will be correct if it is equivalent to the answer. Looking at the main operators ($\supset$ and $\vee$) it is clear that the formulae will be 0 less often than 1. So we begin by assigning 0 to each.

$$\begin{array}{cccc|ccc}
\sim & A & \supset & B & A & \vee & B \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 3 & 1 & 2 & 2 & 1 & 2
\end{array}$$

Each formula = 0 iff $A = 0$ and $B = 0$. So the formulae are equivalent, and the student's translation is correct.

(This shows that both formulae = 0 on row 4, and = 1 on the other rows of their truth table.)

Example 3: Prove the Associative Law of Conjunction i.e. $p \;\&\; (q \;\&\; r) \Leftrightarrow (p \;\&\; q) \;\&\; r$

Here we choose to assign 1 rather than 0 to the formulae (why?)

$$\begin{array}{ccccc|ccccc}
p & \& & (q & \& & r) & (p & \& & q) & \& & r \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 3 & 2 & 3 & 3 & 2 & 3 & 1 & 2
\end{array}$$

Each = 1 iff $p=1, q=1, r=1$. So they are equivalent. (This shows the formulae = 1 on row 1 and = 0 on the other rows of their table.)

In relation to the third method, although splitting may occur, it is often possible by judicious choice to make the first case yield an assignment set for which the formulae differ in value: in such a case you can stop work immediately as this means the formulae are not equivalent. Note that if one formula splits and the other doesn't, this tells us straight away that they are not equivalent.

EXERCISE 5.4

1. For each of the following, use MAV to determine whether the first formula tautologically implies the second.

   (a) $\sim p \supset q$; $\sim q \supset p$
   (b) $p \;\&\; (q \vee r)$; $(p \vee q) \;\&\; r$
   (c) $p \;\&\; (p \equiv q)$; $q \vee r$

2. For each of the following use MAV to determine whether the formulae are tautologically equivalent. If not tautologically equivalent, state an assignment of values to the PVs where the formulae differ in value.

   (a) $\sim(p \vee q)$; $\sim p \;\&\; \sim q$
   (b) $p \vee (q \vee r)$; $(p \vee q) \vee r$
   (c) $p \supset (q \supset r)$; $(p \supset q) \supset (p \supset r)$
   (d) $p \supset (q \;\&\; r)$; $(p \supset q) \supset r$
   (e) $(p \;\&\; q) \supset r$; $p \supset (q \supset r)$
   *(f) $(p \vee q) \;\&\; (q \;\&\; r)$; $(p \;\&\; q) \;\&\; (q \vee r)$

3. Given that A, B and C are indifferent propositions, determine which of the following pairs are necessarily equivalent.

   (a) $A \;\&\; \sim B$; $\sim(A \supset B)$
   (b) $A \supset (B \vee C)$; $(A \supset B) \vee C$
   *(c) $(C \;\&\; A) \supset B$; $\sim B \vee (C \;\&\; A)$

5.5 TESTING ARGUMENTS

As regards argument-*forms*, a *counterexample* is an assignment of values to its PVs which makes each premise = 1 and the conclusion = 0. Recall from §4.6 that an argument-form is *valid* iff there is no row in its truth table with such an assignment. The MAV test for validity of argument-forms begins by assuming such a counter-row does exist, and then seeing whether this assumption generates a contradiction.

PL-argument-form	MAV test
valid	assigning 1 to each premise and 0 to the conclusion → full closure
invalid	the above test fails to give full closure

We set out the argument-form as for a truth table but assume we are on a counter-row. The original assignments to the premises and the conclusion are each numbered as step 1. As soon as an alternative is found which will remain open we copy the assignments of the PVs across to the matrix to indicate the counterexample and thus establish invalidity.

Example 1: Test the following argument-form for validity.

$$\begin{array}{l} p \,\&\, q \\ q \supset r \\ \hline \therefore\ r \end{array}$$

p	q	r	p & q	$q \supset r$	$\therefore r$
			1 1 1	$\underline{1}$ 1 1	0
			2 1 2	— 1 3	1

×

Full closure.

∴ Valid

Notice from the above example that values may be copied across a row e.g., the value of q in the second premise was copied from the value of q in the first. This is allowed because on a truth table the PVs must have the values of the matrix row. The second thing to note about the above example is that care should be taken about which propositional-form to work on next e.g., here the first premise was worked before the second to avoid splitting. Lastly, note that if full closure occurs we leave the matrix row blank.

Example 2: Test the following argument-form for validity.

$$p \vee q,\ q \supset r\ /\ \therefore r$$

p	q	r	$p \vee q$	$q \supset r$	$\therefore r$
1	0	0	1 1 0	0 1 0	0
			3 1 —	2 1 —	1

Counterexample as shown.

∴ Invalid.

As regards ***arguments***, a counterexample is a ***possible*** world in which the premises are true and the conclusion is false. An argument is valid iff it has no counterexample. From §4.4 and our earlier work, the following tests may be stated. Here "no cases close" means open cases occur for ***all*** assignments to the propositional letters.

Argument	MAV test on explicit PL-translation
PC-valid	assigning 1 to premises and 0 to conclusion → full closure
PC-invalid	assigning 1 to premises and 0 to conclusion → no cases close
PC-indeterminate	assigning 1 to premises and 0 to conclusion → Just some cases close

Argument	MAPV test on any PL-translation
valid	assigning 1 to premises and 0 to conclusion → full closure
invalid	the above test fails to give full closure

In addition, a check can be made on whether the premises are *inconsistent*, by assigning 1 to each and seeing whether full closure results.

Example 3: Symbolize the following argument, then assess its validity.

> Either Cathy or Donna will be there. Alan will be there if and only if either Bill or Cathy will be. Now, Donna won't be there. So both Alan and Cathy will be there.

Using an obvious dictionary, this symbolizes as:

$C \vee D,\ A \equiv (B \vee C),\ {\sim}D\ /\ \therefore A \,\&\, C$

A	B	C	D	C	$\vee$	D	A	$\equiv$	$(B$	$\vee$	$C)$	$\sim$	D	A	$\&$	C	
				1	1	0	0	1	0	0	0	1	0	0	0	1	Full
				3	1	—	—	1	6	5	$\underline{6}$	1	2	4	1	—	closure
												x					∴ Valid

As remarked earlier, private solutions can be sped up by omitting step numbers, but it is always helpful to enter the initial assignments in a different colour and to underline copied values. With invalid arguments that split, thoughtful selection can arrange for a counterexample to occur in the first alternative, and work can immediately stop. Since truth trees (to be discussed in the next chapter) provide a simpler method of dealing with splitting, we have focussed our attention on examples that need not split.

NOTES

Our presentation of MAV is a modified and extended version of the procedure developed by Malcolm Rennie for use in propositional and modal logic (see *Logic: Theory and Practice* §§1.8–1.10). Variations of MAV go under many names e.g., "abbreviated truth-table method", "reductio ad absurdum test", "trial-and-error method". That the method of setting out is by no means standardized can be seen by consulting recent treatments by other authors e.g., Tapscott's *Elementary Applied Symbolic Logic* Ch. 14, and Bradley and Swartz's *Possible Worlds* §5.10.

EXERCISE 5.5

1. Use MAV to assess the following argument-forms for validity. Where invalid, state a counterexample.

   (a) $q \supset p,\ q\ /\ \therefore p$
   (b) $q,\ p \supset q\ /\ \therefore p$
   (c) $p \supset q,\ q \supset r,\ r \supset s,\ s \supset t\ /\ \therefore p \supset t$
   (d) $p \vee (q \,\&\, r),\ q \,\&\, {\sim}p\ /\ \therefore r \supset p$
   (e) $p \supset {\sim}(q \supset r),\ {\sim}(r \supset (s \not\equiv t))\ /\ \therefore {\sim}r \vee t$

2. The following arguments are represented by their explicit PL-translations. Use MAV to assess them as PC-valid, PC-invalid or PC-indeterminate.

   (a) $A \vee B,\ {\sim}B\ /\ \therefore A$
   (b) $A \supset B,\ {\sim}A\ /\ \therefore {\sim}B$
   (c) $A \supset (B \,\&\, C),\ {\sim}D \supset A,\ {\sim}B\ /\ \therefore D \,\&\, {\sim}A$
   (d) ${\sim}(D \supset E),\ C \equiv A,\ {\sim}(D \,\&\, B)\ /\ \therefore A \vee B$

3. Given that *A, B, C, D* and *E* in Question 2 are indifferent propositions, state which

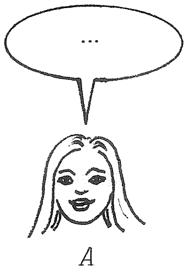
arguments are valid and which are invalid. Where invalid state a counterexample.

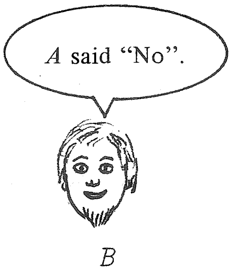
4. (a) Use MAV to show that the following argument is valid.
   $\sim(A \vee (B \vee C))$, $A \vee D$, $\sim D$ / $\therefore \sim(B \supset C)$
   (b) Is the argument sound? (Hint: Use MAV to test the premises for consistency).

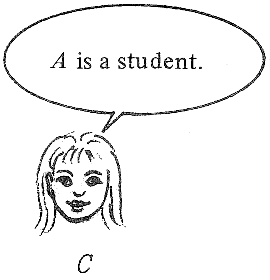
5. Symbolize the following arguments using the suggested letters, then use MAPV to assess their validity. Where invalid state a counterexample.
   *(a) If the world is heading for a crisis then, unless people try to be more loving the human race is doomed. Obviously the world is heading for a crisis. From all this it follows that the human race will try to be more loving but will be doomed. (*C, L, D*)
   (b) Meditation will be effective only if concentration is. Now for concentration to be effective it is necessary both that good posture be attained and that the mind be relaxed. Consequently, if either good posture is not attained or the mind is not relaxed then meditation will not be effective. (*M, C, P, R*)

Puzzle 5 In a certain college, students always lie and lecturers always tell the truth. *A, B* and *C* are three people from this college. When asked if she is a student, *A* replied (with a "Yes" or "No"). *B* and *C* then commented as shown.

How many of these three people are students?







5.6 SUMMARY

The Method of Assigning Values (MAV) often allows faster evaluations in PC than the truth table method. An RAA technique, it begins by assigning a value to the main operator(s) of the formula(e) involved and determining by means of rules whether this generates a contradiction in all cases (i.e. "*full closure*"). *Assignment Rules* allow values of operands to be deduced from the values of their operators: some of these generate a pair of alternatives (this is called "*splitting*"). The *Efficiency Rule* advises us to avoid using a splitting rule unless there is no choice. The *Copy Rule* permits values to be copied to all other occurrences of the same letter in the same alternative. The *Resolution Rule* allows the value of a symbol to be resolved from previous values, by making use of implications from our operator definitions. The *Closure Rule* allows an alternative to be closed if (i) both 1 and 0 are assigned to different occurrences of the same expression, or (ii) an operator value is inconsistent with the values of its operand(s) by violating the operator definition. A case remains *"open"* until closed. The *Completion Rule* allows us to stop, once all cases close or it is obvious that one case will remain open.

The Method of Assigning Possible-Values (MAPV) adds the following *Subatomic Closure Rule:* close any alternative with an inconsistent assignment of values to its propositional constants due to the internal nature of the dictionary propositions. Like possible-truth tables, MAPV goes beyond PC in allowing internal analysis of atomic propositions.

MAV and MAPV solutions can be made more intelligible to others by underlining copy steps and numbering other steps.

PL-forms may be assessed by MAV as follows: tautology (assign 0 → full closure); contradiction (assign 1 → full closure); otherwise contingency.

Propositions may be assessed by applying MAV to their explicit PL-translation as follows: tautology (assign 0 → full closure); PC-contradiction (assign 1 → full closure); otherwise PC-indeterminacy. Propositions may also be assessed by applying MAPV to any of their PL-translations as follows: necessary truth (assign 0 → full closure); contradiction (assign 1 → full closure); otherwise contingency.

As regards *relations*, MAPV is usually not very efficient unless just one relation is being checked. To determine whether α tautologically/necessarily implies β, use MAV/MAPV to determine whether $\alpha \supset \beta$ is a tautology/necessary-truth. Equivalence is best tested as follows: make an educated guess as to which value α and β will least frequently exhibit; assign this value to both and apply MAV (or MAPV); α and β are equivalent iff the open cases remaining match exactly.)

PL-argument-forms may be assessed by MAV as follows: valid (assign 1 to premises and 0 to conclusion → full closure); otherwise invalid. *Arguments* may be assessed by applying MAV to their explicit PL-translation as follows: PC-valid/PC-invalid/PC-indeterminate according as assigning 1 to premises and 0 to conclusion → full closure/no cases close/just some cases close. Arguments may also be assessed by MAPV as follows: valid (assign 1 to premises and 0 to conclusion → full closure); otherwise invalid.

6

Truth Trees

6.1 INTRODUCTION

We now move on to one of the most enjoyable parts of logic: *truth trees.* Though trees are delightful, they perform the same jobs as tables or MAV; so the less adventurous reader may object to learning yet another testing procedure for propositional logic. Two main reasons may be given in reply to such an objection. The short-term reason is that, like MAV, trees often give *quicker* solutions than tables do, particularly if *4 or more propositional letters* are involved; moreover, trees provide a *simple means of dealing with* cases which, by *splitting;* prove awkward for MAV. The long-term reason for studying truth trees is even more important: when we graduate from PC to the more powerful QT (Quantification Theory), PC-trees may be *easily extended into QT-trees* (currently the best testing procedure available in QT); on the other hand, truth tables and MAV have very limited application in QT.

Like MAV, truth trees are most useful when testing a single property or relation and when testing validity. After laying down general rules for making our trees "grow", we will use the tree method to test propositions, relations and arguments. The possible-truth tree method will also be introduced, and an advanced tree method employing resolution rules will be briefly treated.

6.2 GENERAL RULES

Replacement Rules :

Unlike botanical trees, truth trees grow *downwards* from an initial formula or set of formulae. New growth results from use of *replacement rules*, whereby an expression may be replaced by another that is *equivalent* to it. The *replacing expression* is written below the *replaced expression,* the latter being *ticked* off to remind us that we no longer need refer to it since its information is now available from the replacing expression. As an example of a simple tree, consider:

$$\begin{array}{ll} \surd & \sim\sim(p \,\&\, q) \\ & \quad p \,\&\, q \end{array}$$

Here, $\sim\sim(p \,\&\, q)$ has been replaced by $p \,\&\, q$, using the DN equivalence $\sim\sim\alpha \Leftrightarrow \alpha$ and employing our practical concession which allows outer-most brackets to be omitted. Thus our first replacement rule is merely a reformulation of the Law of Double Negation.

Rule 1: √ $\sim\sim\alpha$

α

Now that you have the general idea let us proceed with the other replacement rules. In each of these we let α and β be any wffs.

Rule 2: √ $\alpha \,\&\, \beta$

α

β

This is based on the idea that $\alpha \,\&\, \beta$ is true iff both α and β are each individually true. We may extend this to

Rule 2′: √ $\alpha_1 \,\&\, \alpha_2 \,\&\, \ldots \,\&\, \alpha_n$

α_1

α_2

.

.

.

α_n

This amounts to the general result that a conjunction is true iff all its conjuncts are true.

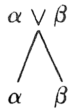
Example: √ $p \,\&\, \sim q \,\&\, (r \supset p)$

p

$\sim q$

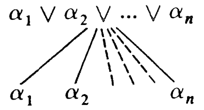
$(r \supset p)$

Rule 3: √

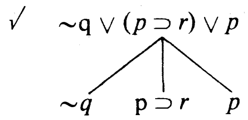


This is our first example of "branching". A disjunction is true iff at least one of its disjuncts is true. On the left branch we treat the case where α is true: on the right branch we consider the case where β is true. Note that we leave the question open whether β is true on the left and whether α is true on the right. The branch symbol $\wedge$ thus functions like a wedge (written upside down and bigger). From rules 2 and 3 we see that moving along the one stem or branch involves conjunction whereas the branching process itself is a matter of disjunction. Rule 3 may be generalised to

Rule 3′: √

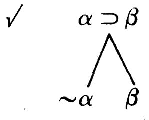


Example:

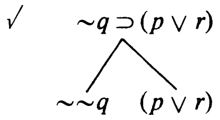


Though equivalent to splitting in MAV, branching in trees is more economical since there is no need to make a second copy of the previous information (it can be readily accessed by climbing up the tree). The next rule, which is based on the equivalence $\alpha \supset \beta \Leftrightarrow \sim \alpha \vee \beta$, corresponds to the MAV assignment rule $\alpha \supset \beta = 1 \rightarrow \alpha = 0$ or $\beta = 1$.

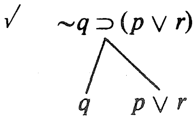
Rule 4:



Example:

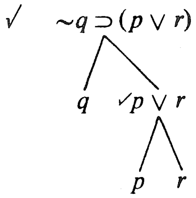


In practice, to save a bit of writing, we would usually write instead



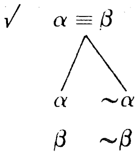
On the left branch we have mentally applied the rule of Double Negation and on the right hand branch the brackets have been dropped since this leads to no ambiguity.

You may be wondering if rule 3 may now be applied to the right branch. Yes, this is correct. By doing so we obtain



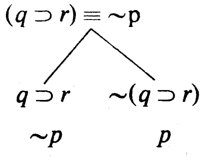
You can now appreciate how application of several rules may lead to quite large trees.

Rule 5:



This depends on the fact that α is materially equivalent to β iff they have the same truth value, i.e. they are either both true (left branch) or both false (right branch).

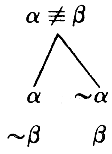
Example: √



Note that in the right branch the brackets must be retained and that DN has been used to get p.

It is important to realize that since the replacement rules are equivalence rules for the *whole formula* being replaced, in any particular case it is the *main operator* which determines the replacement rule to be selected. For instance, in the above example the main operator of $(q \supset r) \equiv \sim p$ is $\equiv$, not $\supset$; so in replacing this formula we must use the rule for $\alpha \equiv \beta$ (i.e. Rule 5), not the rule for $\alpha \supset \beta$ (i.e. Rule 4).

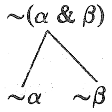
Rule 6: √



This follows from the fact that $\alpha \not\equiv \beta$ is true iff α and β have opposite truth values.

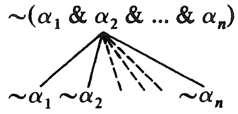
The rest of our replacement rules are for negated expressions.

Rule 7: √



A conjunction is false iff at least one of its conjuncts is false. This allows the rule to be extended to

Rule 7′: √



Rule 8: √ $\sim(\alpha \vee \beta)$

$\sim\alpha$

$\sim\beta$

A disjunction is false iff all its disjuncts are false. Hence this may be extended to the following rule, 8′.

Rule 8′: $\surd$ $\sim(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n)$

$\sim\alpha_1$

$\sim\alpha_2$

.

.

.

$\sim\alpha_n$

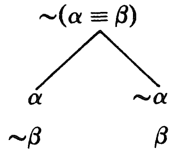
Rule 9: $\surd$ $\sim(\alpha \supset \beta)$

α

$\sim\beta$

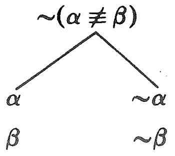
As the truth table for $\alpha \supset \beta$ shows, the expression is false on and only on the second row, i.e. when α is true and β is false.

Rule 10: $\surd$



This follows immediately from Rule 6 and the fact that $\sim(\alpha \equiv \beta)$ is equivalent to $\alpha \not\equiv \beta$.

Rule 11: $\surd$



This follows from Rule 5 since $\sim(\alpha \not\equiv \beta)$ is equivalent (by Double Negation) to $\alpha \equiv \beta$.

If you have previously studied MAV, all of the basic (unprimed) replacement rules should have seemed familiar to you. The reason for this is that each of these is analogous to an MAV assignment rule. You might like to check for yourself the fact that the second through the twelfth assignment rules of §5.2 correspond, in order, to the tree replacement rules 1, 2, 7, 3, 8, 4, 9, 5, 10, 6 and 11. The MAV efficiency rule also has the following tree analogue:

Efficiency Rule: Don't branch until you have to.

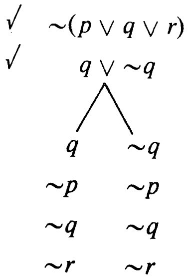
While it is not essential to obey this rule, it usually saves us work if we do. What it means is that if we have a choice between using a rule that doesn't branch (i.e. one of rules 1, 2, 2′, 8, 8′, 9) and one that does (i.e. one of rules 3, 3′, 4, 5, 6, 7, 7′, 10, 11) we should use the non-branching rule to use first. If forced to branch, the replacement rules for $\equiv$ and $\not\equiv$ are usually best to apply first (where relevant) since they provide two entries on each alternative.

Example: Suppose we have the two expressions below on a common branch of our tree.

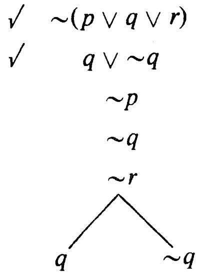
$\sim(p \vee q \vee r)$

$q \vee \sim q$

Applying rules 3 and 8′ in that order gives:



However, using our efficiency rule to apply 8′ before 3 yields the shorter result



Clearly this saves duplication.

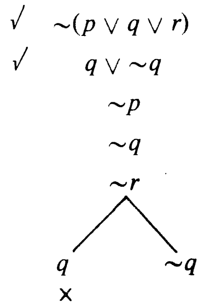
For obvious reasons, the MAV Copy Rule has no analogous tree rule. Tree resolution rules analogous to the MAV Resolution Rules may be specified, but since branching does not engender as much work as splitting they are not as urgently required for labour-saving as their MAV counterparts are: we will treat them as an advanced technique and postpone their discussion till later in the chapter. Both the MAV Closure Rule and Completion Rule do have corresponding tree rules, so let us look at these now. We will use the word *"path"* to describe a continuous, downward route starting at the top of the tree. A tree has more than one path iff it branches.

Closure Rule: Close a path (by writing × underneath it) as soon as a contradiction of the form

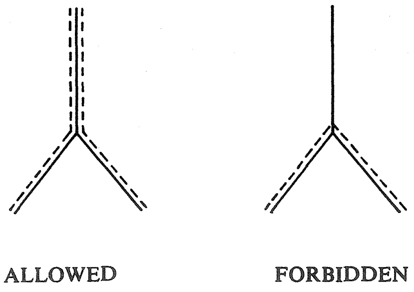
$$\begin{array}{c}\alpha\\ \cdot\\ \cdot\\ \cdot\\ \sim\alpha\end{array}$$

occurs in it (where α is any wff).

As an illustration, the left hand path in the above example should be closed since it effectively contains both q and $\sim q$ in it. This would be written



Because of our replacement rules, you are entitled to climb from any position on a branch all the way up to the start of the tree and count all the information passed on the way as being true of your particular path. However you are *not* allowed to climb up one branch and go down another: the information on one branch need not be true of another branch. The dotted lines in the diagram below indicate allowed and forbidden paths on a skeleton tree.



For instance, the right hand path in the example above contains $\sim q$ and the left contains q: but this does not constitute a contradiction since the elements are on different paths. Hence it would be a "forbidden path error" to close the right path at this stage.

Two further points are worth noting about the tree closure rule. Firstly, α and $\sim\alpha$ may occur *anywhere* on the path. For example,

$$\begin{array}{c} \vdots \\ \sim(p \,\&\, q) \\ \vdots \\ p \,\&\, q \\ \vdots \end{array}$$

may be closed. Secondly α must be a *whole element* on the path, not just a part. For example,

$$p \supset q$$

$$\vdots$$

$$\sim p$$

may NOT be closed by regarding the "p" in "$p \supset q$" as "α" and "$\sim p$" as "$\sim \alpha$".

Earlier in this text we defined a propositional *letter* to be either a propositional variable or a propositional constant. We now define an *elementary wff* of PL to be either a propositional letter or a negated propositional letter. For instance, each of the following is an elementary wff: "p", "$\sim q$", "A", "$\sim B$". The tree Completion Rule may now be stated as follows.

Completion Rule: Keep going until (i) there is at least one open path with unticked elements consisting only of elementary wffs,
or (ii) all paths close.

Note that alternatives (i) and (ii) are mutually exclusive. This is because once a path has been reduced to elementary wffs it can be simplified no further; so if a contradiction has not occurred by this stage it will not occur at all, i.e. the path will never close.

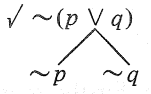
NOTES

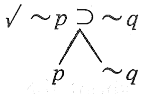
Trees for PC and QT are comparatively recent inventions. They derive from the semantic tableaux technique first set out by Evert Beth in 1955. Our method of setting out trees is a modified version of that used by M. K. Rennie.

Some authors use "branch" the way we use "path". Our usage treats a tree consisting only of a trunk as having one path but no branches.

EXERCISE 6.2

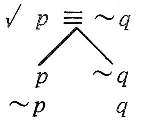
1. Which of the following demonstrate correct use of replacement rules?

(a)


(b)


(c) $\surd\ (p \ \& \ q) \equiv r$
p
q

(d) $\surd \sim(p \ \& \sim q)$
$\sim p$
q

(e)


(f) $\surd \sim(p \vee \sim q \vee r)$
$\sim p$
q
$\sim r$

6.3 TESTING PROPOSITIONS

Any truth tree begins with a trunk containing one or more expressions. The immediate aim of growing the tree is *to determine whether these original expressions involve a contradiction.* Since the tree paths are simply attempts to satisfy the original expressions, if there exists at least one open path that will never close this constitutes a way of making these expressions true i.e. the original expressions do not involve a contradiction. On the other hand, if all the paths close there is no way of making the original expressions true and so they must involve a contradiction. To sum up, *the original expressions in a tree* *involve a contradiction iff they generate a contradiction on all paths.* This general result underlies all of our tree tests.

Before using trees to test propositions let's look at PL-forms. From the above result we can immediately write down the procedure for identifying a self-contradictory form. The only original expression in the tree will be the form to be tested, and it will be a contradiction iff it produces a contradiction on all paths i.e. all paths close. Since a tautology always = 1, its negation must always = 0; so a form is a tautology iff its negation is a contradiction. Any PL-form which is not a tautology or a contradiction must be contingent. These considerations allow us to specify the following tree tests for PL-forms.

PL-form	truth tree test
tautology	negate the form and apply the rules → all paths close
contradiction	affirm the form and apply the rules → all paths close
contingency	each of the above tests fails to give full closure

Example 1: Show that $p \mathbin{\&} {\sim}p$ is a contradiction.

Solution:

$$\begin{array}{ll} \surd & p \mathbin{\&} {\sim}p \\ & p \\ & {\sim}p \\ & \times \end{array}$$

This problem is about as easy as you can get; it needed only one application of Rule 2 before closure, and you should have no difficulty in following the solution provided above. However, sometimes trees get very long and complicated; and to make them easier to follow we introduce a convention to help explain the steps. On the left of our tree we number the lines consecutively so that they can be referred to easily. This gives

$$\begin{array}{lll} \surd & 1. & p \mathbin{\&} {\sim}p \\ & 2. & p \\ & 3. & {\sim}p \\ & & \times \end{array}$$

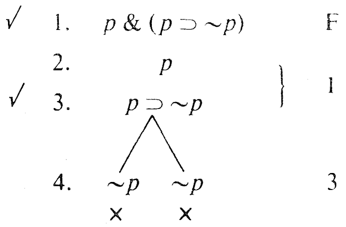
On the right of our tree we provide a *justification column.* First of all we identify the formula to be tested for a contradiction by writing "F" (for "Formula"). Secondly, in applying the Replacement Rules we quote the line used at each step. This gives the final form of our tree.

$$\begin{array}{llll} \surd & 1. & p \mathbin{\&} {\sim}p & \text{F} \\ & \begin{array}{l} 2. \\ 3. \end{array} & \left.\begin{array}{l} p \\ {\sim}p \end{array}\right\} & 1 \\ & & \times & \end{array}$$

Even though it involves more writing, it is best to clarify your moves in this way, at least until there are so many branches that continuing the justification becomes cumbersome. Your tree will then be more intelligible not only to someone else, but also to yourself if you return to it after leaving it for a while.

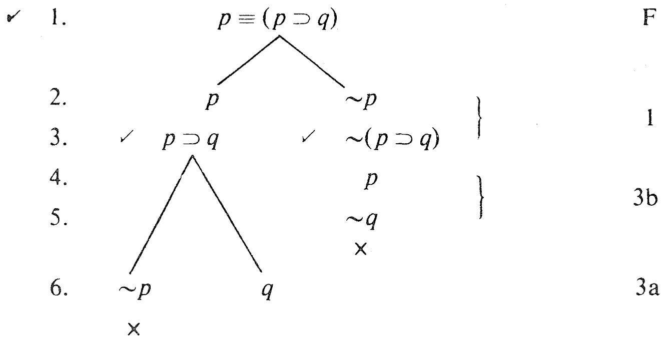
Example 2: Test to see if $p \mathbin{\&} (p \supset \sim p)$ is a contradiction.

Solution:



All paths close. Therefore $p \mathbin{\&} (p \supset \sim p)$ is a contradiction. (Replacement Rules 2 and 4 were used in this solution.)

Example 3: Test to see if $p \equiv (p \supset q)$ is a contradiction.

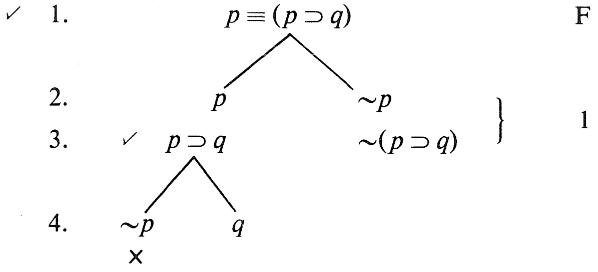


There is an unclosed path on the completed tree. Therefore $p \equiv (p \supset q)$ is *not* a contradiction.

Notes: 1. When there are 2 or more paths for a particular line, the items on the paths should be ticked rather than the line number.

2. In order to refer to items on different paths but the same line, the paths may be ordered alphabetically from left to right. Thus "3a" refers to the item on line 3 and in the leftmost path; "3b" identifies the element on line 3 in the path which is second from the left, etc.

3. Our completion rule informs us that we may stop work as soon as we find an open path with unticked elements consisting only of elementary wffs. Thus in the example above, if we had taken step 6 straight after step 3 we would have been able to obtain the following solution.



The path second from the left cannot close. Therefore $p \equiv (p \supset q)$ is not a contradiction.

When testing for *tautologies* we begin by *negating the formula*, and to indicate this write "NF" instead of "F" in the justification column. Remember that *if the formula has a dyadic main operator but no outer brackets, it must be bracketed for negation.* The formula is a tautology iff all paths close.

Example 4: Show that $p \vee \sim p$ is a tautology.

✓	1.	$\sim(p \vee \sim p)$	NF
	2.	$\sim p$	1
	3.	p	
		×	

Here we used Rule 8 (and also DN for line 3).

Example 5: Test to see if $p \supset (q \supset p)$ is a tautology.

✓	1.	$\sim(p \supset (q \supset p))$	NF
	2.	p	1
✓	3.	$\sim(q \supset p)$	
	4.	q	3
	5.	$\sim p$	
		×	

$\therefore \quad p \supset (q \supset p)$ is a tautology.

Here Rule 9 was used twice.

Example 6: Test $p \supset q . \equiv . \sim q \supset \sim p$ for tautologyhood.

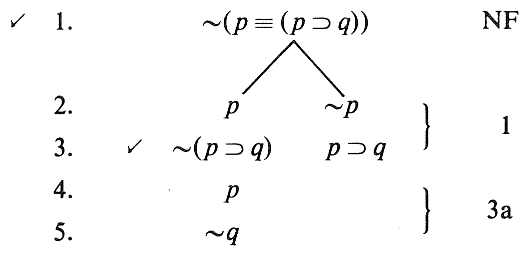
✓	1.		$\sim(p \supset q . \equiv . \sim q \supset \sim p)$					NF
	2.	✓	$p \supset q$		✓	$\sim(p \supset q)$		1
	3.	✓	$\sim(\sim q \supset \sim p)$		✓	$\sim q \supset \sim p$		
	4.		$\sim q$			p		3a, 2b
	5.		p			$\sim q$		
	6.		$\sim p$	q		q	$\sim p$	2a, 3b
			×	×		×	×	

$\therefore \quad p \supset q . \equiv . \sim q \supset \sim p$ is a tautology.

In going through this solution you should have noticed:

1. the use of Rules 10, 9, 4 and DN;
2. space saving by having paths share the same line and justification space.

Example 7: Test to see if $p \equiv (p \supset q)$ is a tautology.



The left path will not close.

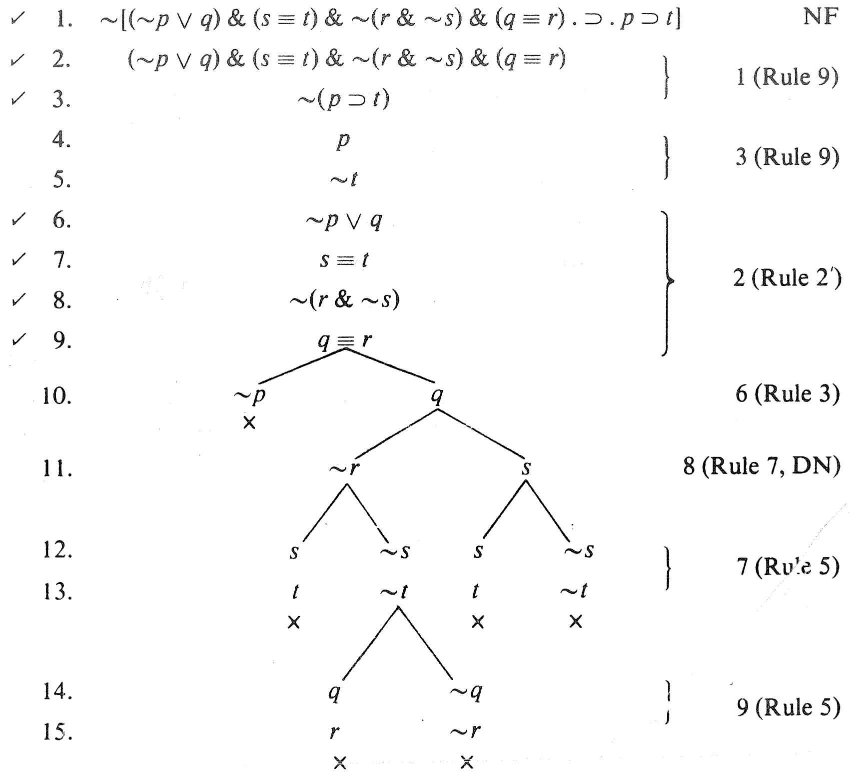
∴ $p \equiv (p \supset q)$ is *not* a tautology.

You may have notice that the form tested in the latest example is the same one tested in the third example on contradictions. In both cases the test proved negative, i.e. $p \equiv (p \supset q)$ is neither a contradiction nor a tautology. What is it then? It must be a contingency.

Once you understand and learn the rules, all truth trees are easy. Like truth tables, some are just longer (rather than harder) than others. Let's look at a longer example now.

Longer Example:

Test to see if $(\sim p \vee q) \,\&\, (s \equiv t) \,\&\, \sim(r \,\&\, \sim s) \,\&\, (q \equiv r) \,.\supset .\, p \supset t$ is a tautology.



∴ the formula is a tautology.

Note carefully that *the replacing expression must always be added to every open path stemming from the replaced expression,* e.g., both pairs of paths on lines 12 and 13 are required. This is important.

You will no doubt agree that the above tree solution is shorter than the 32 row truth table solution. In the tree, the Replacement Rules used have been named to make it easier for you to follow. However you need not bother to do this in your solutions. Neither should you learn which number denotes which replacement rule. All that is needed is that you understand and be able to write down each Replacement Rule (not the name). The best way to do this is to mentally derive them from our knowledge of the operators, and to have plenty of practice with them.

EXERCISE 6.3A

1. Use trees to see which of the following are contradictions.

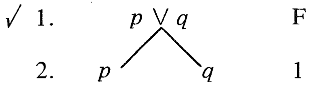
    (a) $\sim(p \supset p)$
    (b) $p \not\equiv p$
    (c) $p \equiv \sim(p \,\&\, p)$
    (d) $p \,\&\, \sim(q \supset p)$
    (e) $\sim p \,\&\, \sim(p \supset q)$
    (f) $p \supset (\sim p \vee q)$
    (g) $p \equiv (q \not\equiv r)$
    (h) $(p \supset q) \,\&\, (q \supset r) \,.\equiv.\, (p \,\&\, \sim r)$
    (i) $\sim [(q \not\equiv (r \,\&\, \sim r)) \vee (p \supset (q \supset p))]$
    (j) $p \,\&\, (\sim r \equiv s) \,\&\, \sim(q \vee \sim(p \supset q))$
    (k) $(p \supset \sim(q \vee s)) \,\&\, (q \not\equiv \sim r) \,.\supset.\, p \supset \sim r$
    (l) $\sim p \,\&\, \sim q \,\&\, \sim r \,\&\, \sim s \,\&\, \sim t \,.\not\equiv.\, \sim(p \vee q \vee r \vee s \vee t)$

2. Use trees to see which of the following are tautologies.

    (a) $p \supset p$
    (b) $\sim(p \supset \sim p)$
    (c) $\sim p \supset (q \supset p)$
    (d) $p \supset (p \vee q)$
    (e) $(p \supset q) \supset (q \supset p)$
    (f) $(p \supset q) \,\&\, (q \supset p) \,.\equiv.\, p \equiv q$
    (g) $p \equiv (q \vee r) \,.\supset.\, p \supset r$
    (h) $(p \supset (q \vee r)) \,\&\, (p \equiv \sim q) \,.\supset.\, \sim p$
    (i) $\sim [r \equiv (q \vee p) \,.\supset.\, (p \,\&\, r)]$
    (j) $p \vee (q \,\&\, r) \,.\equiv.\, (p \vee q) \,\&\, (p \vee r)$
    (k) $(p \equiv r) \supset (\sim s \vee q) \,.\&.\, \sim r \supset (s \supset \sim p)$
    (l) $(p \supset s) \,\&\, \sim s \,\&\, (q \equiv r) \,.\supset.\, r \vee (p \supset (\sim q \,\&\, s))$
    (m) $p \,\&\, q \,\&\, r \,\&\, s \,\&\, t \,.\supset.\, p \vee q \vee r \vee s \vee t$

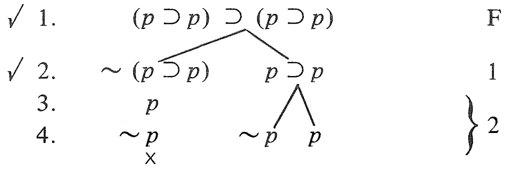
The standard tree method for discovering a contingency is to perform two separate tests, one for a tautology and one for a contradiction, and show that it fails both. This is what we did earlier with the formula of Examples 3 and 7. But there is a superior method, whereby contingencies may be detected from a single tree: to explain this it will be helpful to introduce some new terminology. When we discussed truth tables it was convenient to speak about a particular "matrix row", but trees don't have such rows, so we need a more general term to convey the idea of a particular set of truth value assignments: we will use the term "model". *A model for a PL-wff is an assignment of truth values to each of its propositional letters.* A formula with n propositional letters will have 2^n models e.g., $p \vee q$ has four models: p=1, q=1; p=1, q=0; p=0, q=1; p=0, q=0. To save writing let's agree that several letters may be strung together with "=" signs to indicate assignment of the same value to each e.g., the assignments p=1, q=1 and p=0, q=0 may be abbreviated to p=q=1 and p=q=0 respectively. Given any PL-form we may now classify it as a tautology iff it = 1 in all (its) models, as a contradiction iff it = 0 in all models, and as a contingency iff it = 1 in some models and = 0 in others.

To apply this theory to truth trees we need to examine the connection between paths and models. *If a completed path is open, it corresponds to one or more models,* as found from the elementary wffs on the path (a letter is assigned 0 if it has a ∼ in front). Consider the following tree.



On the left path p=1 but the value of q is unspecified, so this path covers the two models p=q=1 and p=1, q=0. On the right path q=1 and the value of p is optional, so this path corresponds to the two models p=q=1 and p=0, q=1. Notice that both paths contain the model p=q=1. In general, *the same model may be included in different paths.* Although the paths above cover 3 models, they do not cover the model p=q=0: this model has been eliminated by the Replacement Rule, thus revealing that $p \vee q$ = 0 in this model. From this one tree then, we have been able to determine that $p \vee q$ = 1 in some models and = 0 in others i.e. it is contingent.

Since there is no model in which a contradiction = 1, and paths are closed only if they contain a contradiction, it follows that *any closed path corresponds to no model at all.* Consider for instance the following tree.



The left path is closed and hence contains no model (there is no assignment where p=1 and p=0). The open middle path corresponds to the model p=0, and the open right path to the model p=1. Moreover, these two models are the only models for the formula. Hence the formula = 1 in all models and consequently is a tautology.

You may have guessed by now that scanning open paths of a completed tree for models will immediately enable a form to be classified, regardless of whether it is a tautology, contradiction or contingency. This is true even if we begin with a negated form, as it is easy to show. These considerations may be formulated into a technique which we call the *"One-Tree Method"* for classifying forms, because it tests for the three types of form in a single tree. The method may be summarized as follows.

Method:
1. Make an educated guess as to which value the form will least frequently exhibit and assign this value to it (by affirming or negating the form)
2. Complete the tree

3. If all paths close, the form is a contradiction or tautology according as the form was originally affirmed or negated
4. If some paths remain open then:
   if the open paths cover all models the form is a tautology or contradiction according as the form was originally affirmed or negated; if the open paths do not cover all models the form is contingent.

The first step in the method is one of efficiency: if in doubt, simply affirm the formula and the method will work anyway. When the form turns out to be a contingency it is best to clearly demonstrate this by specifying a model where the form = 1 and another where it = 0. Since locating two such models is enough to prove contingency, it is not necessary to complete the tree if these models can be derived from an incomplete tree: this allows the following modification to the above method:

> As soon as you find one permanently open path, and one eliminated model, stop the tree: the form is contingent.

Please turn back to Examples 3 and 7 and you will see how the One-Tree Method detects the contingency from either tree. In Example 3, there is only one open path and this contains just the model $p=q=1$; so the form = 0 in the other three models. With Example 7, which begins by negating the form, the open path indicates that the form = 0 in the model $p=1$, $q=0$; from the two paths of the incomplete tree it is obvious that the model $p=q=1$ has already been eliminated, so the form = 1 in this model. From either tree then, we are able to list two models in which the form has different values, and so either tree can be used to establish contingency. If desired, the values from the two models listed may be substituted into the form to check the result.

When using trees to classify propositions we may adopt the standard tree method as follows.

Proposition	truth tree test on explicit PL-translation
tautology	negate the proposition → all paths close
PC-contradiction	affirm the proposition → all paths close
PC-indeterminacy	each of the above tests fails to give full closure

Alternatively, the One-Tree Method may be used on the explicit PL-translation (just replace the terms "form", "contradiction" and "contingency" with "proposition", "PC-contradiction" and "PC-indeterminacy").

As with possible-truth tables and MAPV, PC-indeterminacies may be further resolved into necessary truths, contradictions and contingencies by means of *possible-truth trees*. To construct these we augment the standard tree technique with the following rule.

Subatomic Closure Rule: Close any path which has an inconsistent assignment of values to its propositional constants due to the internal nature of the dictionary propositions.

Once this second closure rule has been applied, any permanently open path must contain at least one *possible* model and consequently will represent at least one possible world: such a path may be called a *"possible path"*. Possible-truth trees thus sort propositions as follows.

Proposition	Possible-truth tree test on any PL-translation
Necessary Truth	negate the proposition → full closure
Contradiction	affirm the proposition → full closure
Contingency	each of the above tests fails to give full closure

Alternatively, the One-Tree Method may be used on any PL-translation (just replace "form", "tree" and "tautology" with "proposition", "possible-truth tree" and "necessary truth"). Like possible-truth tables and MAPV, possible-truth trees have no application to PL-forms.

Example 8: Use possible-truth trees to assess the modal status of the following proposition.

If Sue is neither angry nor upset then she is not very angry.

Using A = Sue is angry
U = Sue is upset
V = Sue is very angry

this translates as $\sim(A \vee U) \supset \sim V$

Since our intuitions suggest the proposition is a necessary truth, we begin by negating it.

$$\begin{array}{rll}
\checkmark\ 1. & \sim[\sim(A \vee U) \supset \sim V] & \text{NF} \\
\checkmark\ 2. & \sim(A \vee U) & \rbrace\ 1 \\
3. & V & \\
4. & \sim A & \rbrace\ 2 \\
5. & \sim U & \\
 & \times &
\end{array}$$

Closure results from the fact that any path asserting both V and $\sim A$ is impossible (there is no possible world in which Sue is very angry but not angry). So the proposition is a necessary truth.

NOTES

Some authors use the term ' model" in a more restricted sense to mean a set of assignments for which the formula = 1. Our usage allows models in which the formula = 0: we call these "countermodels" for the formula. A model in which the formula = 1 we call a "satisfying model" for the formula.

Both the One-Tree Method and Possible-truth Trees are, so far as we know, new. A technique analogous to the One-Tree Method may be adopted for MAV and MAPV: for instance, if you look back to Example 3 of §5.3, contingencyhood may be deduced from the first test alone since the form is therein shown to = 0 only in the model p=1, q=0 and hence it = 1 in all other models.

EXERCISE 6.3B

1. For each of the following paths, list the models that are included. In each case, the elements listed are the only elementary wffs on a completed path stemming from a PL-form whose PVs are p, q and r.

(a) $\sim p$	(b) p	(c) $\sim r$	(d) p
q	q	$\sim p$	q
$\sim r$			$\sim p$

2. Use the One-Tree Method to classify each of the following forms as a tautology, contradiction or contingency. For any contingency, list two models where the form has different values.

(a) $p \supset \sim p$
(b) $q \supset (p \supset q)$
(c) $p \supset (p \supset q)$
(d) $\sim p \ \& \ \sim(p \supset q)$

(e) $(p \vee q) \equiv q$
(f) $(p \mathbin{\&} {\sim}q) \not\equiv r$
(g) ${\sim}(p \mathbin{\&} q \mathbin{\&} r \mathbin{\&} s)$

3. Symbolize the following propositions using the suggested letters, then use the One-Tree Method to classify each as a tautology, PC-contradiction or PC-indeterminacy.
   (a) Although Plato is Greek it's not true to say that either Socrates is not Greek or Plato is Greek. (P, S)
   (b) Unless it rains, Tom will go to both the football and the cricket. (R, F, C)
   (c) If the sea is blue then it's calm only if it's a calm, blue sea. (B, C)

4. Given that R, F, C as defined for Question 3 (b) are indifferent, give a more precise classification of the proposition there.

5. Symbolize the following propositions using the suggested letters, then use possible-truth trees to classify each as a necessary truth, contradiction or contingency.
   (a) Sue is joyful only if she is not miserable. (J, M)
   (b) Sue is not joyful only if she is miserable. (J, M)
   (c) Although the number is not divisible by 2 it is both positive and even. (D, P, E)
   (d) Tarski writes logic books but is not an author, only if he is an author but doesn't write logic books. (L, A)

6.4 TESTING RELATIONS

Although truth trees and possible-truth trees may be used to sort relations into the categories discussed for tables in §3.7, in most cases they are less efficient than tables. We limit our interest here to implication and equivalence.

Since α tautologically *implies* β iff $\alpha \supset \beta$ is a tautology, to test whether $\alpha \Rightarrow \beta$ we simply apply the tree test to determine whether $\alpha \supset \beta$ is a tautology i.e. negate $\alpha \supset \beta$ and see whether all paths close.

Example 1: Determine whether $p \mathbin{\&} q$ tautologically implies q.

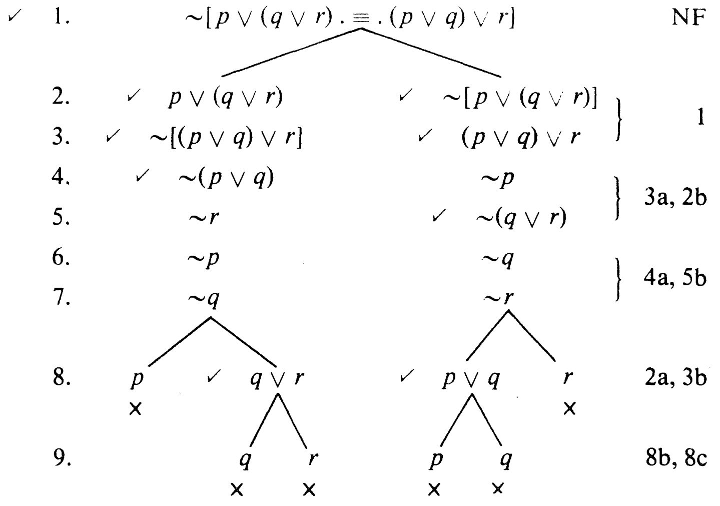
✓	1.	${\sim}((p \mathbin{\&} q) \supset q)$		NF
✓	2.	$p \mathbin{\&} q$		1
	3.	${\sim}q$		
	4.	p	}	2
	5.	q		
		×		

$\therefore (p \mathbin{\&} q) \supset q$ is a tautology i.e. $p \mathbin{\&} q \Rightarrow q$.

Similarly, to test whether proposition α necessarily implies proposition β, we apply the possible-truth tree test to determine whether $\alpha \supset \beta$ is a necessary truth.

The straightforward way of testing whether α is tautologically/necessarily *equivalent* to β is to test whether $\alpha \equiv \beta$ is a tautology/necessary truth i.e. negate $\alpha \equiv \beta$ and see whether all paths close in the truth/possible-truth tree.

Example 2: Test to see if $p \vee (q \vee r)$ is tautologically equivalent to $(p \vee q) \vee r$.



All paths close. $\therefore\ p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$.

NOTES

A Two-Tree Method can be developed which enables several relations to be tested at once from the trees for two formulae by comparing remaining models. But unless the number of propositional letters is very large, tables are more efficient.

EXERCISE 6.4

1. For each of the pairs below, use trees to determine whether the first member tautologically implies the second.

(a) $p \,\&\, q \ ; \ p$
(b) $p \vee q \ ; \ p$
(c) $\sim(p \vee q \vee r \vee s \vee t) \ ; \ (p \supset t) \supset (r \supset s)$

2. For each of the pairs below, use trees to determine whether the members are tautologically equivalent.

(a) $p \supset q \ ; \ q \supset p$
(b) $p \,\&\, q \ ; \ p \,\&\, (p \supset q)$
(c) $p \supset (q \vee r \vee s \vee t \vee p) \ ; \ (r \supset s) \supset (\sim s \supset \sim r)$

6.5 TESTING ARGUMENTS

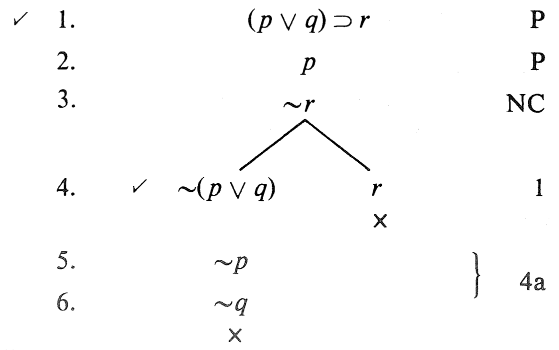
As regards argument-*forms*, a counterexample is a model in which each premise = 1 and the conclusion = 0. An argument-form is valid iff it has no counterexample. The tree test for validity begins by assuming a counterexample does exist and seeing whether this assumption generates a contradiction i.e. whether all paths close.

PL-argument-form	Truth tree test
valid	affirm each premise and negate the conclusion → all paths close
invalid	the above test fails to make all paths close.

To indicate the form being tested is that of an argument we write "P" beside each premise and "NC" beside the negated conclusion. With tables the conclusion is not negated, but with trees it is: don't confuse the two procedures. Since any permanently open path contains at least one model we can read off a counterexample from it: this is best specified in the same way as for tables.

Example 1: Test the following argument-form for validity.

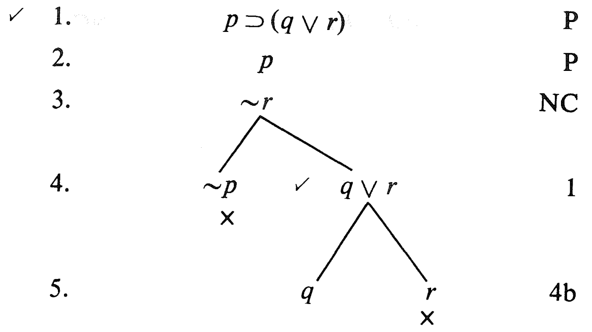
$$\begin{array}{ll} & (p \lor q) \supset r \\ & p \\ \hline \therefore & r \end{array}$$



All paths close. ∴ the argument-form is valid.

Example 2: Test the following argument-form for validity.

$$\begin{array}{ll} & p \supset (q \lor r) \\ & p \\ \hline \therefore & r \end{array}$$



An open path remains. ∴ the argument-form is invalid.

Counterexample:

p	q	r
1	1	0

As regards *arguments,* a counterexample is a possible world in which all the premises are true and the conclusion is false. An argument is valid iff it has no counterexample. Although the set of all models covers the set of all possible worlds, without an internal analysis of the dictionary propositions we are unable to determine whether a particular model is possible. From earlier work, the following tests may now be stated.

Argument	Truth tree test on explicit PL-translation
PC-valid	affirm each premise and negate the conclusion → all paths close
PC-invalid	the above test fails to eliminate any model
PC-indeterminate	the above test eliminates just some models

Example 3: Test the following argument for validity.

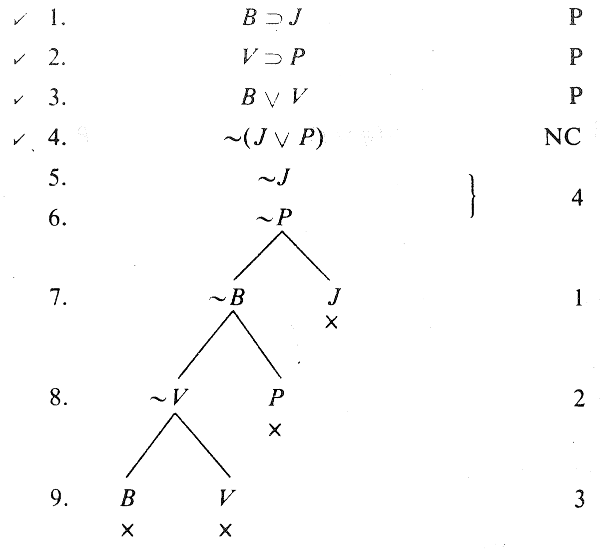
> If Bernard is my brother then so is John, and Vince is my brother only if Paul is. Now at least one of Bernard and Vince is my brother; so either John is my brother or Paul is.

Dictionary:
B = Bernard is my brother
J = John is my brother
V = Vince is my brother
P = Paul is my brother

Translation:

$$\begin{array}{l} B \supset J \\ V \supset P \\ \underline{B \vee V} \\ \therefore\ J \vee P \end{array}$$

Tree Solution:



All paths close. ∴ the argument is valid.

Example 4: Test the following argument for validity.

> If Lee learns astronomy he gains satisfaction, and if he teaches astronomy he gains money. Hence whenever he teaches astronomy he doesn't learn astronomy.

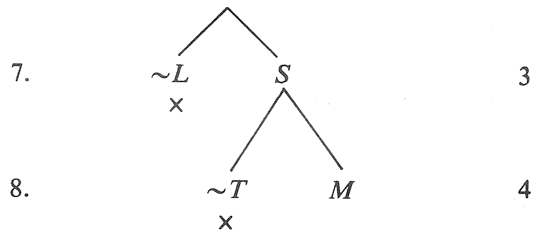
Dictionary:
L = Lee learns astronomy
S = Lee gains satisfaction
T = Lee teaches astronomy
M = Lee gains money

Translation:

$$\frac{(L \supset S)\ \&\ (T \supset M)}{\therefore\ T \supset \sim L}$$

Tree Solution:

✓	1.	$(L \supset S)\ \&\ (T \supset M)$	P
✓	2.	$\sim(T \supset \sim L)$	NC
✓	3.	$L \supset S$	1
✓	4.	$T \supset M$	
	5.	T	2
	6.	L	



The open path corresponds to the model L=S=T=M=1.
The other models have been eliminated.
So the argument is PC-indeterminate.

With *possible-truth trees,* the Subatomic Closure Rule ensures that any open path remaining must correspond to at least one possible world and hence provide a counterexample. This allows arguments to be classified as follows.

Argument	Possible-truth tree test on any PL-translation
valid	affirm each premise and negate the conclusion → all paths close
invalid	the above test → at least one open path

For instance, with Example 4 the surviving model from the standard tree test, viz. L=S=T=M=1, describes a world where Lee learns and teaches astronomy, and gains satisfaction and money. Clearly such a world is possible, so the argument may now be pronounced invalid and the counterexample may be specified in the usual way as:

L	S	T	M
1	1	1	1

That's enough examples. A few points are worth mentioning though, before you work through the following exercise. Remember that the same model may be catered for on more than one open path, and that one open path may supply more than one model. Although argument-forms and arguments may have several counterexamples, one is enough to establish invalidity. Sometimes you may find that a tree closes before you have ticked off all the premises: this happens sometimes because arguments may have *redundant premises* (i.e. extra premises which are not really required to establish the conclusion). Trees may also be used for detecting inconsistent premises: list just the premises and apply the rules; if all paths close the premises are inconsistent.

EXERCISE 6.5

1. Use truth trees to show that the following argument-forms are valid. (These forms will be used in the chapter on natural deduction, so we have included their full and abbreviated names for later reference.)

(a) $p \,\&\, q \;/\; \therefore p$	Simplification (Simp)
(b) $p, q \;/\; \therefore p \,\&\, q$	Conjunction (Conj)
(c) $p \vee q,\ \sim p \;/\; \therefore q$	Denying a Disjunct (DD)
(d) $p \;/\; \therefore p \vee q$	Addition (Add)
(e) $p \supset q, p \;/\; \therefore q$	Affirming the Antecedent (AA)
(f) $p \supset q,\ \sim q \;/\; \therefore \sim p$	Denying the Consequent (DC)
(g) $p \supset q, q \supset r \;/\; \therefore p \supset r$	Chain Argument (Ch Ar)
(h) $p \vee q, p \supset r, q \supset s \;/\; \therefore r \vee s$	Complex Constructive Dilemma (CCD)
(i) $\sim\sim p \;/\; \therefore p$	Double Negation (DN)
(j) $\sim p \,\&\, \sim q \;/\; \therefore \sim(p \vee q)$	DeMorgan (DeM)

2. Use trees to test the following argument-forms for validity. Where invalid, state a counterexample.
   (a) $\sim p \;/\; \therefore p \supset \sim p$
   (b) $p \supset q, q \;/\; \therefore p$
   (c) $\sim(p \,\&\, q), p \;/\; \therefore \sim q$
   (d) $\sim(p \supset q),\ p \not\equiv q \;/\; \therefore q \,\&\, \sim p$
   (e) $p \supset \sim q, q \supset \sim p \;/\; \therefore \sim(p \vee q)$
   (f) $p \supset (q \,\&\, r),\ \sim q \vee \sim r \;/\; \therefore \sim p$
   (g) $p \supset q, r \supset q, p \vee r \;/\; \therefore q$
   (h) $p \supset q, p \supset r, p \;/\; \therefore q \vee r$
   (i) $p \supset q, r \supset q, q \;/\; \therefore p \vee r$
   (j) $p \supset q, r \supset s,\ \sim q \vee \sim s \;/\; \therefore \sim p \vee \sim r$
   (k) $p \equiv (q \vee r),\ \sim q \equiv s, s \;/\; \therefore r \supset p$
   (l) $\sim p \,\&\, (r \supset s),\ \sim(s \vee q), q \supset p \;/\; \therefore r \not\equiv p$
   (m) $(p \supset q) \,\&\, (s \supset t),\ \sim(r \,\&\, \sim s) \;/\; \therefore (q \supset r) \supset \sim(p \,\&\, \sim t)$

3–5 Answer Exercise 5.5, Questions 2, 3 and 4, using truth trees instead of MAV.

6. Symbolize and test the following arguments by possible-truth trees. Provide a dictionary in each case, and where invalid state a counterexample.
   (a) If the Labor Government stays in power, inflation will continue. Since the Labor Government will not stay in power it follows that inflation will discontinue.
   (b) The colour can't be both red and blue. It's not red. So it must be blue.

(c) The teacher will resign if his job becomes intolerable. Now if the students become even more ill-mannered, his job will be intolerable unless he exerts greater discipline. The students are becoming more ill-mannered. Hence unless he exerts greater discipline the teacher will resign.

(d) It's not true to say that Peter's getting a job is sufficient to prevent Quentin from getting a job. However, both Peter and Quentin will get a job only if someone gets the sack. But someone's getting the sack guarantees that Peter will miss out on a job. From all this we may deduce that Quentin won't get a job.

(e) Exactly one of Sharon and Maurice will go on the trip. Sharon will not go on the trip only if she decides that she would rather spend her time studying logic. She won't decide that she would rather spend her time on logic unless logic is very interesting; and Maurice will go on the trip if and only if logic is very interesting. So either Sharon will decide that she would rather spend her time studying logic, or she will not go on the trip, or logic is not very interesting.

(f) Either you are a logician or you are just an ordinary person; and if you are a male then you are strong. Now you are just an ordinary person unless you have been enlightened. Hence *if* you are neither strong nor enlightened *then* either you are a logician or a male or you have not been enlightened.

(g) If I don't get fresh ideas you're not going to get any more problems to do. If you don't get any more problems you won't have enough work to do. If you haven't got enough work to do you'll either make up some more problems for yourself or you'll have a loaf. If you have a loaf you'll stop improving. Hence, for you to improve it's necessary that either you make up some more problems for yourself or I get some fresh ideas.

*(h) If I believe that God exists I am a theist, whereas I am an atheist if I believe that God doesn't exist. However, I am neither a theist nor an atheist provided that I am an agnostic. Therefore I am an agnostic only in case that both I do not believe that God exists and I do not believe that God doesn't exist.

*(i) At least one of Alan, Bert, Cindy, Darlene and Eartha survived the car accident. Exactly one of Alan and Bert survived. Moreover, Cindy will survive if and only if either Darlene or Eartha does. Now at least one of Alan, Cindy and Eartha survived. Nevertheless, Darlene and Bert both survive or neither of them does. From all this we may conclude that Alan, Cindy and Eartha survive but the others don't.

(j) If Special Relativity is correct this neutrino cannot move at the speed of light unless it has zero rest mass. Now if this neutrino doesn't move at the speed of light, momentum will not be conserved. Either momentum is conserved or classical physics needs further revision. No further revision of classical physics is needed however, so we may deduce that either this neutrino has zero rest mass or Special Relativity is incorrect.

*(k) Answer Exercise 4.4, Question 2(h), using possible-truth trees instead of possible-truth tables.

6.6 TREES WITH RESOLUTION

Like MAV, truth trees are adequate for evaluation in PC but can be shortened by augmenting the rules with a resolution rule.

Resolution Rule: Where possible, a value may be assigned to a symbol by resolving it with respect to values already assigned on the same path.

Since resolution sub-rules were discussed in §5.2, instead of repeating them here we'll look at just a couple to see how they carry over from MAV to trees. The presence of α as a whole element on a path is equivalent to the assignment $\alpha = 1$. The presence of $\sim\alpha$ as a whole element on a path is equivalent to the assignment $\alpha = 0$. When using a resolution rule we tick the resolved expression and place the appropriate resolution(s) on the paths stemming from it. In the justification column we note the line number (and path letter if applicable) of the resolved expression, and bracket the assignment(s) with respect to which the expression has been resolved.

Let's see how Example 2 from §6.5 comes out with resolution.

√ 1. $p \supset (q \vee r)$ P
2. p P
3. $\sim r$ NC
√ 4. $q \vee r$ 1 (p=1)
5. q 4 (r=0)

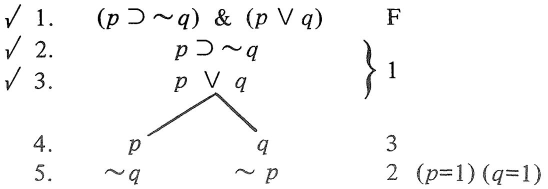
Invalid. Counterexample:

p	q	r
1	1	0

Steps 4 and 5 used the following respective sub-rules:

α	$\supset$	β	α	$\vee$	β
1	1	1	1	1	0
		*	*		

As a more complicated case, consider the following One-Tree test with resolution.


√ 1. $(p \supset \sim q)$ & $(p \vee q)$ F
√ 2. $p \supset \sim q$
√ 3. $p \vee q$ } 1
4. p q 3
5. $\sim q$ $\sim p$ 2 (p=1) (q=1)

Form = 1 in just 2 models ∴ contingent.

The resolution for step 5a used the following sub-rule:

α	$\supset$	β
1	1	1
		*

The resolution for step 5b used the following sub-rules:

$\sim$	α	α	$\supset$	β
0	1	0	1	0
*		*		

The Resolution Rule may be applied to both truth trees and possible-truth trees, in all their applications.

NOTES

The resolution technique discussed here is a modified and slightly less powerful version of the method developed by Ian Hinckfuss for use with both PC-trees and QT-trees.

EXERCISE 6.6

1. Answer Exercise 6.3B, Question 2, using resolution.
2. Answer Exercise 6.5, Question 1, using resolution.

*3. Symbolize the following arguments using the dictionary supplied, and test them for validity using possible-truth trees with resolution. Where invalid state a counterexample.

(a) Population growth will halt only if either the birth rate is reduced or the death rate is increased. Unless population growth does halt, the human race is doomed. For the birth rate to be reduced it is both necessary and sufficient that people act responsibly. Unfortunately however, people will not act responsibly. Nevertheless, the population growth will halt. Hence either the death rate will increase or the human race is doomed or perhaps both. (H = Population growth will halt; R = The birth rate is reduced; I = The death rate is increased; D = The human race is doomed; A = People act responsibly.)

(b) If we believe that reality is simply as we have been taught then our experience is limited by our descriptive framework. Unless we make proper use of our will, our experience of reality will be limited by our language and we will fail to activate one of our rings of power. Although we are luminous beings, we do not make proper use of our will. Hence, we will fail to activate one of our rings of power only if either we believe that reality is just as we've been taught or we are not luminous beings. (B = We believe that reality is simply as we have been taught; E = Our experience of reality is limited by our linguistic framework; W = We make proper use of our will; F = We will fail to activate one of our rings of power; L = We are luminous beings.)

(c) Fromm is correct only if both (i) love is inexhaustible
and (ii) one cannot love oneself if one is incapable of loving others.

Unless Freud was wrong however, love is not inexhaustible. Freud was wrong provided that one's incapability of loving others implies one's incapability of loving oneself. Obviously then, Fromm and Freud cannot both be correct. (C = Fromm is correct; I = Love is inexhaustible; S = One is incapable of loving oneself; O = One is incapable of loving others; W = Freud was wrong).

(d) I would believe in God if someone showed me an argument which proved God's existence; however nobody has shown me any such argument. Having a direct experience of God would be sufficient for me to believe in God, provided that I knew what the experience meant. I would know what the experience meant if my mind was ready. I do not yet believe in God. From this it may be deduced that either I have not had direct experience of God or my mind is not ready. (B = I believe in God; A = Someone shows me an argument which proves the existence of God; E = I have a direct experience of God; K = I know what the experience means; R = My mind is ready.)

Puzzle 6 For each of the following, try to work your way from top to bottom, at each step changing no more than one letter to produce an English word.

(a) BASIS
– – – – –
TASKS

(b) PATH
– – – –
– – – –
LAWS

(c) CITES
– – – – –
– – – – –
– – – – –
– – – – –
MODAL

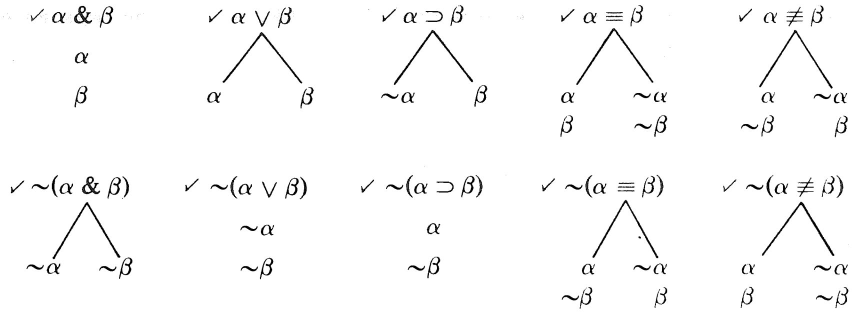
(d) THINK
– – – – –
– – – – –
– – – – –
– – – – –
– – – – –
PROVE

(e) TREES
– – – – –
– – – – –
– – – – –
– – – – –
– – – – –
– – – – –
PLOTS
– – – – –
– – – – –
– – – – –
– – – – –
– – – – –
TRUTH

6.7 SUMMARY

Truth trees provide a PC-evaluation method, particularly useful when several propositional letters are involved, and extendable to QT.

Replacement Rules: ✓ $\sim\sim\alpha$
α (Double Negation (DN))



DN may be applied at any stage without notification. With the other rules, tick the replaced expression and quote the line used.

Extended replacement rules:

✓ $\alpha_1 \,\&\, \alpha_2 \,\&\, \dots \,\&\, \alpha_n$

α_1

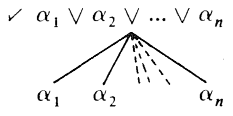
α_2

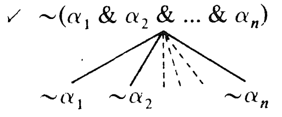
.

.

.

α_n





✓ $\sim(\alpha_1 \vee \alpha_2 \vee \dots \vee \alpha_n)$

$\sim\alpha_1$

$\sim\alpha_2$

.

.

.

$\sim\alpha_n$

Efficiency Rule: Don't branch until you have to.

Closure Rule: Close a path (with an ×) as soon as a contradiction of the form

α

.

.

.

$\sim\alpha$

occurs in it.

Completion Rule: Keep going until (i) there is at least one open path with unticked elements consisting only of elementary wffs,
or (ii) all paths close.

An *elementary wff* of PL is either a propositional letter or a negated propositional letter. *Paths* go all the way to the top of the tree. When an expression is replaced, the replacing expression should be included on all open paths stemming from the ticked expression. In setting out a tree, a justification column is advised (until it becomes unwieldy) to help explain the steps.

Possible-truth trees add the following *Subatomic Closure Rule:* Close any path which has an inconsistent assignment of values to its propositional constants due to the internal nature of the dictionary propositions.

The original expressions in a tree involve a contradiction iff they generate a contradiction on all paths.

PL-forms may be assessed by truth trees as follows: tautology (negate the form (NF) → all paths close); contradiction (affirm the form (F) → all paths close; contingency (each of the above tests → an open path).

A *model* for a PL-wff is an assignment of truth values to each of its propositional letters. A closed path contains no models but a permanently open path contains at least one model. A PL-form is a tautology/contradiction/contingency according as it = 1 in all/none

/just some models. The *One-Tree Method* for PL-forms starts by affirming or negating the form: if all paths close it is a contradiction or tautology by the usual test; as soon as one permanently open path and one eliminated model are found the form may be declared contingent.

Propositions may be assessed by applying truth trees to their explicit PL-translation as follows: tautology (NF → all paths close); PC-contradiction (F → all paths close); otherwise PC-indeterminacy. The One-Tree Method may also be adapted for use here. Propositions may also be assessed by applying possible-truth trees to any of their PL-translations as follows: necessary truth (NF → all paths close); contradiction (F → all paths close); otherwise contingency. The One-Tree Method may also be adapted for use here.

As regards *relations*, trees are not as generally efficient as tables. Relations may be tested by applying a tree test to the appropriate formula. For example, $\alpha \Rightarrow \beta$ iff T: $\alpha \supset \beta$, and $\alpha \Leftrightarrow \beta$ iff T: $\alpha \equiv \beta$.

PL-argument-forms may be assessed by trees as follows: valid (affirm premises (P) and negate conclusion (NC) → all paths close); otherwise invalid. *Arguments* may be assessed by applying truth trees to their explicit PL-translation as follows: PC-valid/PC-invalid/PC-indeterminate according as affirming premises and negating conclusion → all paths close/no models eliminated/just some models eliminated. Arguments may also be assessed by possible-truth trees on any of their PL-translations as follows: valid (affirm premises and negate conclusion → all paths close); otherwise invalid. Inconsistent premises may be detected by just affirming the premises and seeing whether all paths close.

A more sophisticated tree method with resolution adds the following *Resolution Rule:* A value may be assigned to a symbol by resolving it with respect to values already assigned on the same path. The sub-rules for resolution are discussed in the MAV section §5.2.

7 Realistic Analysis Of Arguments

7.1 INTRODUCTION

The techniques you have learned for analysing arguments are quite powerful, but they should not be applied blindly to arguments met in everyday situations. Frequently a certain amount of common sense, or even uncommon sense, is necessary to appropriately assess everyday arguments. In this chapter we discuss a number of issues which should be borne in mind when conducting such argument analysis.

If we maintain our formal notion of the term "argument", an argument really has only two aspects to be investigated; one *logical* (do the premises support the conclusion in the manner claimed?); the other *material* (are the premises, as a matter of fact, true?). However,. if we widen the notion of argument to include persuasion-aimed speech acts, a third, *rhetorical* dimension is brought in. This third aspect is often quite important from the human point of view and is worthy of serious study, but we have space here to offer only the following few pieces of advice in this regard.

First, actually *listen* to what the speaker has to say; try not to let any personal prejudice you might have against the speaker bias your analysis of his actual argument. Secondly, determine what the speaker is trying to *do* with his words e.g., arguing for a conclusion, offering an explanation, shocking you out of a blinkered approach to the topic, emotionally manipulating you. Try not to be tricked by the speaker's rhetorical skill into letting your emotions carry you to irrational courses of action. Thirdly, it is sometimes helpful to *empathize* with the speaker: by imagining yourself in his shoes, having gone through his experiences, it will help you to understand why he has adopted his position. Finally, Edward de Bono's *PMI analysis* can be of use (i.e. isolate the plus, minus and just plain interesting points in the speaker's approach). Matters such as these may be pursued further in texts on informal logic. We now return to the analysis of arguments as entities which, once clarified, can be studied independently of their proposers.

In the next section we consider the problem of determining just what the argument is, then look at various ways in which the argument might be modified. In the following sections we discuss a recently developed method for deciding, in particular cases, whether a PL-translation is adequate for evaluation purposes.

7.2 IDENTIFYING AND MODIFYING ARGUMENTS

Given an argument, there are three main questions that need to be answered in order to identify precisely what the argument is:

What type of argument is it?
What is the conclusion?
What are the premises?

The type of argument is *deductive* or *inductive* according to whether the conclusion is claimed to follow certainly, or just probably, from the premises. The methods discussed in this book apply only to deductive arguments. If you have some doubt about locating the conclusion, or you find the conclusion unclear (e.g., because of an ambiguous term), you have two options open to you: if the proposer of the argument is available you can ask him to *clarify* what he meant; or you can state clearly what you take the conclusion to be and proceed on that basis. Sometimes the proposer of the argument leaves the conclusion *tacit* (i.e. unstated), expecting you to infer the "obvious" implication of his premises, but this is fairly uncommon. It is quite common however for one or more *premises* to be left unstated, usually because the proposer assumes they would be common knowledge to his audience. Technically, any argument with a tacit premise is called an *enthymeme*. Here is a simple example.

Socrates is a man.
Therefore Socrates is mortal. (1)

This enthymeme has the tacit premise "All men are mortal": this premise must be taken into account when assessing the argument for validity. When unclear about any premise (stated or tacit) we again have the two standard options: request clarification from the proposer; or state our interpretation and proceed.

Once the premises have been clearly identified, they may be checked for factual errors. If any *relevant* premise (i.e. a premise whose presence is necessary to establish the validity of the argument) is found to be false, the argument may immediately be assessed as unsound. If the argument has no relevant factual error, and is proven to be valid (e.g., by the techniques of propositional logic), then it may be assessed as sound and the conclusion accepted as true. If the argument is found to be invalid then it is unsound and the conclusion is in doubt. The standard way to demonstrate invalidity is to produce a counterexample, either by a logical technique such as tables or simply by using one's imagination. Another effective way to challenge the validity claim of an argument is to produce a *counterargument* i.e. an argument which is obviously invalid but which has the same relevant logical form as the argument in question. Consider the following argument for example:

If he's a communist he favours strike action.
He does favour strike action.
So he's a communist. (2)

An appropriate counterargument would be:

If you're a communist then you breathe.
You do breathe.
So you're a communist. (3)

Sometimes, what appears to be a counterargument turns out not to be because the original argument differs in the way its component propositions are modally related e.g., (3) above is not a counter to argument (5) of §4.6: production of an apparent counterargument is still useful however since it can help to reveal relevant subatomic structures in the original argument.

Sometimes, people confuse the procedure of testing an argument with that of testing an argument-form: this can lead to arguments being prematurely branded invalid. It will help to guard against this confusion if we draw a distinction between the terms "counterexample" and "countermodel" as they apply to arguments translated into PL. A *counterexample* is a possible world in which the premises are true and the conclusion is false. A *countermodel* is an assignment of truth values to the propositional letters which makes each premise true and the conclusion false. Countermodels sometimes describe possible worlds: in this case they describe counterexamples. However, some countermodels are impossible: these fail to describe any possible world and hence do not describe a counterexample. In short, a *countermodel provides a counterexample iff the countermodel is possible.*

Once a countermodel for an argument has been detected, there are two questions that ought to be asked about it:

> Is the countermodel possible?
> Is the countermodel factual?

If the countermodel is seen to be impossible (either by a more powerful logic or "natural intuition") then it fails to provide a counterexample; in this case, unless some other possible countermodel is detected, the argument may be pronounced valid. If the countermodel is seen to be possible then it does provide a counterexample, and the argument may be declared invalid. You are already familiar with this sort of analysis from your work on possible-truth tables, MAPV and possible-truth trees.

If a counterexample is found then we know the argument is invalid; but this does not mean that the conclusion must be false. There are plenty of invalid arguments around with true conclusions. However, if we can determine that the countermodel is *factual* (i.e. its truth value assignments are in fact true of the *actual* world) then the actual world constitutes a counterexample and we know that the conclusion is in fact false. Since in such cases the conclusion could have been falsified simply by substituting in the truth values of its components, this application tends to be of use only when the conclusion has a complex molecular structure.

The most useful application of the factual check on the countermodel is when it is determined to be *non-factual*. In such a case the original argument can be modified to produce a new argument with a better chance of validity. This is done by adding an *additional premise* which denies some non-factual assignment in the countermodel and hence eliminates it as a countermodel to the new argument. In some cases this additional premise may be just an undetected tacit premise of the original argument; in other cases it reports an item of information whose relevance was not realized by the proposer of the original argument. If the original argument had only one countermodel then this single move will produce a new argument that must be valid. If the original argument had more than one countermodel and all of these were detected, then one could examine all of these for factuality and try to eliminate all the countermodels by a number of additional factual premises. In practical logical debate, it is more usual for just one countermodel to be produced at a time (usually by imagination): this often results in an original argument being gradually improved by a series of modifications each of which eliminates a countermodel discovered since the previous modification. Sometimes all countermodels are eliminated and hence a valid argument results; sometimes a factual countermodel is found and the conclusion is then known to be false; sometimes countermodels are found whose factuality cannot be determined without further logical or empirical investigation.

To illustrate some of the above ideas, let us consider the following argument.

> If God exists He is all-powerful and perfectly good. If God is willing to prevent evil but cannot do so He is not all-powerful. If God can prevent evil but is unwilling to do so He is not perfectly good. Now evil does exist; moreover, it can exist only if God either cannot, or is unwilling to, prevent it. Hence God does not exist.

It is left as an exercise for you to symbolize this in PL, and then test it with a truth tree, using the following dictionary: G = God exists, A = God is all-powerful, P = God is perfectly good, W = God is willing to prevent evil, C = God can prevent evil, E = Evil exists.

Your tree should have yielded just the following countermodel:

G	A	P	W	C	E
1	1	1	0	0	1

This countermodel describes a world in which an all-powerful, perfectly good God co-exists with evil. The 0's inform us that this God is both unwilling and unable to prevent the evil. Now many Christians would say that this countermodel is not factual e.g., they may say that God *is* capable of preventing evil. Suppose we now add "God can prevent evil" as a premise in the argument. This will obviously dispose of the previous countermodel, since now C must be true. So the new argument is valid. Does this mean we must accept the conclusion i.e. God does not exist? Of course not. It is *soundness*, not just validity, which guarantees the conclusion. To be sound an argument must be valid and have true premises. There are a number of premises in the argument above to which a Christian might take exception e.g., he might reject the third premise on the grounds that God might see that greater good overall will eventuate if He allows the existence of some evil.

Given our notion of argument, there are three ways in which an argument may be modified: *modify the premises; modify the conclusion; modify the claim* as to how strongly the premises support the conclusion. We have looked at the first of these ways. The second and third ways are also important, particularly the second. If it becomes clear that the conclusion is false or that it cannot be deduced from the available evidence then, rather than dispensing with the whole argument, it may be useful to try for a different conclusion on the basis of the same evidence. For instance, in the light of a counterexample to the original argument we may argue for a weaker conclusion that is not open to this counterexample. In some cases, when we regard any further weakening of the conclusion to be unacceptable, we may choose to weaken instead our claim of support e.g., from validity to inductive strength, or from higher to lower inductive probability.

NOTES

Our usage of the term "countermodel" for an assignment of values to the propositional letters which makes each premise = 1 and the conclusion = 0 means that with argument-*forms* (as distinct from arguments) the term "countermodel" is synonymous with "counterexample".

For an in-depth treatment of argument analysis which employes a minimum of symbolization, we suggest that you consult Michael Scriven's *Reasoning* (New York: McGraw-Hill, 1976).

EXERCISE 7.2

1. Write down what you would normally assume to be the tacit premises in the following enthymemes.
   (a) She's a woman. So she must be inferior.
   (b) Tom is not a financial member of the union. Consequently, he will not get the pay rise.
   (c) All enthymemes have tacit premises. So this argument must have a tacit premise.
   (d) To achieve anything really worthwhile requires a good deal of effort. Hence we must work hard if we are to become competent logicians.

2. Each of the following passages contains an argument with the conclusion left unspoken. Identify this tacit conclusion.
   (a) Since you asked for my opinion on whether you should employ Fred I would like to point out that anybody who is lazy or lacking in self discipline is unfit for your job. Fred lacks self-discipline and is very lazy.
   (b) I put it to the Honourable Member of the Government that if he was really interested in maintaining a high standard of education in our country then he would support the view that funds allocated to the education sector should be increased by at least the inflation rate. But here he is suggesting a percentage increase well below the rate of inflation. I leave it to those acquainted with these facts to draw their own inference.

3. Set out an appropriate counterargument for each of the following.
   (a) Dad, if you were a mean old buzzard you wouldn't double my allowance this week. But I know you're not a mean old buzzard. So you will double my allowance this week, won't you?
   (b) Only if you are rich enough to afford it will you buy our latest model limousine. I can see that a fine gentleman like you is rich enough to afford it. So no doubt you will purchase our latest limousine.

4. Can you find a counterargument for the following example? Discuss.

   If the set is empty it has no members.
   The set is not empty.
   So it has members.

5. Mr Theist is trying to prove to Mr Agnostic that God exists. Read the following descriptions of their discussion, and answer the questions indicated (a), (b), (c).

Mr Theist:	"If the universe exists it had a start, and the universe obviously does exist. Now, for something to cause the universe it is both necessary and sufficient that God exists. Thus it follows, my dear Mr Agnostic, that God exists."
Mr Agnostic:	"Not so fast, my dear Mr Theist. I can produce a countermodel to your argument."

   (a) Provide a dictionary using the propositional constants *U, S, C, G* and symbolize Mr Theist's argument. Then use PC to derive a countermodel.

* * *

Mr Theist:	"Nice work my dear Mr Agnostic, but your countermodel is not likely to be consistent with your beliefs. Do you believe that the universe had a start only if something caused it?"

Mr Agnostic: "Well ... Yes."

Mr Theist: "Aha! I can now modify my argument to produce another argument which is unquestionably valid."

(b) Show how Mr Theist can do this, demonstrating clearly that validity results.

* * *

Mr Agnostic: "Well done, my dear Mr Theist. However it is soundness, not just validity, that counts; and the soundness of your argument is still highly questionable."

(c) Discuss briefly what Mr Agnostic means by this, giving an example if you think he has a case.

6. For each of the following arguments invent a counterexample, then modify the conclusion a little to produce a valid argument.
   (a) If he studies Chinese Philosophy he knows of Lao Tzu. If he studies Indian philosophy he knows of Shankara. It is not the case that he knows of both Lao Tzu and Shankara. So he studies neither Chinese nor Indian philosophy.
   (b) If he follows Yogananda then he values both Hinduism and Christianity. If he is an atheist he values neither Hinduism nor Christianity. Now it is a fact that either he follows Yogananda or he is an atheist. From all this it may be inferred that either he doesn't value Hinduism or he doesn't value Christianity.

7. Produce a counterexample to the following argument then weaken the claim of support to make it logically respectable.

   Almost every logic student gets married. So the next student to enroll in logic will get married.

7.3 ADEQUACY OF TRANSLATION

In many cases you will be able to mentally assess the validity of deductive arguments e.g., by naturally deducing the conclusion from the premises (See Ch. 8) or by inventing a counterexample or counterargument. In more complicated cases however, a formal translation and written assessment (e.g., by MAV) may be required. As a matter of practical efficiency the *translation should preferably be as simple as possible.* By and large, we should translate to a language richer than PL only if the validity hinges on more than the use of propositional connectives. Sometimes we can even get away with less than explicit PL-translations (recall §4.2). This general labour-saving approach of using the least detailed translation which is adequate for our purposes is summed up in Quine's *maxim of shallow analysis*: "Where it doesn't itch, don't scratch". If PC is inadequate for evaluation purposes we have three main choices open to us. We might use PC anyway and supplement this with our own intuitions: this was the approach adopted for possible-truth tables, MAPV and possible-truth trees which called for an intuitive sub-atomic analysis. We might use a richer formal language and evaluation system: examples of this approach will be considered in Part Two. Or we might just work in English, using relevant skills developed by our work in formal systems, and adding our own logical insights.

It was noted in §2.4 and §2.6 that, when synonymous or equivalent translations are unavailable in PL, we often resort to implied translations. It is now time to consider in some depth the circumstances under which such implied PL-translations may be confidently regarded as *adequate* for evaluation purposes. For purposes of concise expression, we now introduce symbols for three of the modal relations discussed in §3.7:

$p \leftrightarrow q$ p is necessarily equivalent to q

$p \rightarrow q$ p necessarily implies q

$p \leftarrow q$ p is necessarily implied by q

Since our main use of these symbols will be to discuss the theory of translation adequacy developed by Phillip Staines, we will refer to the symbols collectively as *Staines arrows*. We read "↔", "→", "←" respectively as "is equivalent to", "implies" and "is implied by", with the "necessarily" understood.

It will be convenient to describe a proposition expressed in English as an *English proposition*, and the proposition expressed by its PL-translation as a *PL-proposition*. In the case of dictionary entries, the English proposition and the PL-proposition will be identical, and hence equivalent. For example, if

R = It's raining

then $R \leftrightarrow$ It's raining

However, it will often be the case that the implication holds in one direction only. From our work in §2.4 and §2.6 we are able to determine the appropriate arrow for our standard translations in PL. In the following basic list it is assumed that "and" is used purely conjunctively.

not p $\leftrightarrow$ $\sim p$

(Conj.) p and q $\leftrightarrow$ $p \,\&\, q$

p or q $\rightarrow$ $p \vee q$

p or q $\leftarrow$ $p \not\equiv q$

if p then q $\rightarrow$ $p \supset q$

p iff q $\rightarrow$ $p \equiv q$

p or q or both $\leftrightarrow$ $p \vee q$

p or q but not both $\leftrightarrow$ $p \not\equiv q$

neither p nor q $\leftrightarrow$ $\sim(p \vee q)$

neither p nor q $\leftrightarrow$ $\sim p \,\&\, \sim q$

not both p and q $\leftrightarrow$ $\sim(p \,\&\, q)$

not p or not q $\rightarrow$ $\sim p \vee \sim q$

We give Staines arrows lower priority than either PL operators or English operators. So in the list above we read the arrows as being the main operators.

As was noted in §2.6, "and" is sometimes used in the temporal sense of "and then" (e.g. "She got married and had a baby."), and is sometimes used conditionally (e.g., "Eat lots of lollies and you'll rot your teeth."). Such cases are treated as follows:

(Temporal) p and $q \rightarrow p \,\&\, q$

(Conditional) p and $q \rightarrow p \supset q$

Remember that if the context makes it clear that "or" is being used inclusively then it will be equivalent to "∨"; similarly if the context reveals that "or" is being used exclusively this will be equivalent to "≢".

Look back now to the translation summary in §2.7. In general the phrases there may be treated in a similar manner to the basic phrase above with which they are grouped. An exception must be made for the phrase "is incompatible with" however, because it usually has a modal connotation: we treat is as follows.

p is incompatible with q $\rightarrow$ $\sim(p \,\&\, q)$

You may recall from §2.6 how we dealt with the word "obviously". Various modal phrases may be dealt with as indicated below:

it's necessary that $p \rightarrow p$
we know that $p \rightarrow p$
obviously $p \rightarrow p$
certainly $p \rightarrow p$
it must be that $p \rightarrow p$

it's possible that $p \leftarrow p$
it might be that $p \leftarrow p$

it's impossible that $p \rightarrow \sim p$
it can't be that $p \rightarrow \sim p$

Note that in the context of an argument, modal phrases may sometimes indicate a validity claim rather than being part of the conclusion. In the following argument for instance it seems appropriate to regard the conclusion as being simply "Smith is female". Taking the disjunction in the first premise to be exclusive, the three English propositions will then be equivalent to the three PL-propositions shown.

Smith is either male or female. $\leftrightarrow$ $M \not\equiv F$
Smith is not male. $\leftrightarrow$ $\underline{\sim M}$
(So) Smith (must) be female. $\leftrightarrow$ $\therefore F$

A related case is:

If p then it must be that $q \rightarrow p \supset q$

NOTES

The single horizontal bar in "↔", "→" and "←" reminds us that these denote weaker relations than "⇔", "⇒" and "⇐". Some authors use single-barred arrows for material relations (e.g., "→" for our "⊃") and double-barred arrows for necessary relations (e.g., "⇒" for our "→").

The theoretical basis for the treatment of translation adequacy given in this chapter is provided in Phillip Staines' paper "Some Formal Aspects of the Argument-Symbolisation Relation", *Australian Logic Teachers Journal* Vol 5 No 3 (1981 August).

EXERCISE 7.3

1. Insert the appropriate Staines arrow between the following pairs of propositions. (Dictionary: F = Tom is a farmer; P = Tom is poor; R = Tom is rich; G = Tom is a grocer)

(a)	Although Tom is a farmer he is poor.	$T \& P$
(b)	Although Tom is a farmer he is not poor.	$T \& \sim P$
(c)	Tom could be rich.	R
(d)	If Tom is a grocer he is not a farmer.	$G \supset \sim F$
(e)	If Tom is rich he is a grocer.	$R \supset G$
(f)	If Tom is rich he is a grocer.	$\sim R \vee G$

2. Use the dictionary in Question 1 to translate the following into PL. Then use Staines arrows to show the relationship between the English propositions and the PL-propositions. The English proposition may be abbreviated to the question label i.e. "(a)", "(b)", etc.

   (a) It's not true that Tom is not a farmer.
   (b) It's certain that Tom is not a grocer.
   (c) We know that Tom is rich but not a grocer.
   (d) Tom is not rich and yet he is not poor.
   (e) Either Tom is rich or he is poor.
   (f) Either Tom is a farmer or he is a grocer.

(g) Tom is a farmer only if he is not a grocer.
(h) Tom can't be poor.
(i) It cannot follow that if Tom is rich he is a farmer.
(j) Tom's being a farmer is incompatible with his being rich.

7.4 PRESERVING ARROWS IN COMPLEX CASES

In the previous section, the appropriate Staines arrow holding between the English and the PL proposition could immediately be determined. In more complex cases, where several propositional components are involved, it will be necessary to translate from Engligh to PL in a number of steps. At each step we need to check whether the Staines arrow holding between the English proposition and the previous stage of translation will be preserved at the new stage. In this section we investigate how such a checking procedure may be carried out.

In translating from English we will first replace the dictionary propositions with propositional constants, and then proceed top-down, symbolizing the main operator and then working down through the main operators of sub-formulae. Consider the following example.

If she doesn't diet or exercise then she won't lose weight. (1)

Using: D = She diets
E = She exercises
L = She loses weight

we obtain the equivalent translation:

If not (D or E) then not L (2)

The main operator is now translated to give:

not (D or E) $\supset$ not L (3)

Let us call the Staines arrow that holds between the English proposition and the proposition expressed at this stage of the translation (where the main operator has just been symbolized) the *"initial arrow"*. In this example, since (1) $\leftrightarrow$ (2), and (2) $\rightarrow$ (3) we have (1) $\rightarrow$ (3) i.e. the initial arrow is $\rightarrow$. It should be clear that the arrow holding at a particular stage of translation will be preserved at the next stage if this next stage is obtained simply by replacing components with equivalent ones. Thus, since *not p* $\leftrightarrow$ $\sim p$, the antecedent and consequent of (3) may be translated to give:

$\sim(D \text{ or } E) \supset \sim L$ (4)

Since (1) $\rightarrow$ (3), and (3) $\leftrightarrow$ (4), we have (1) $\rightarrow$ (4). Suppose you're not sure whether the "or" in "D or E" is inclusive or exclusive, so to be on the safe side you translate it by "$\vee$". This yields:

$\sim(D \vee E) \supset \sim L$ (5)

The question now arises as to whether this final step, based on an implied rather than an equivalent translation, will preserve the arrow. That is, does (1) $\rightarrow$ (5)? Our previous work does not tell us. Although we know that D *or* E $\rightarrow$ $D \vee E$ we do not know whether using this implication to translate (4) into (5) will entail that (4) $\rightarrow$ (5). To appreciate that replacing a part with an implied proposition will not necessarily lead to an implied whole, consider the following two propositions:

If I'm a woman then I'm female. (6)
If I'm a human then I'm female. (7)

Although the component "I'm a woman" implies "I'm a human" it is clearly not the case that (6) → (7). What we need then, is some procedure for determining in particular cases whether a non-equivalent translation of a component proposition will preserve the arrow. Fortunately, such a procedure has recently been developed by Phillip Staines, and it is to this that we now turn. Although the procedure is fairly easy to learn and apply, its proof is rather involved and is consequently left for the interested reader to consult in Staines' original work.

In order that the rules of the procedure may be specified concisely we will introduce some new terminology related to *scope*. The operands of any operator are said to constitute the scope of the operator, and any operand parts are said to *lie in the scope* of the operator. Look at the following cases:

$$\sim p \,\&\, q \tag{8}$$

$$\sim\sim p \tag{9}$$

$$\sim(q \vee \sim p) \tag{10}$$

The monadic operator $\sim$ has only one operand. In (8) the scope of $\sim$ is p. In (9) the scope of the first $\sim$ is $\sim p$, and the scope of the second $\sim$ is p. In (10) the scope of the first $\sim$ is $(q \vee \sim p)$, and the scope of the second $\sim$ is p. Note that in (8) p lies in the scope of one $\sim$, in (9) p lies in the scope of two $\sim$s, and in (10) p lies in the scope of two $\sim$s. Dyadic operators have two operands. So any dyadic operator has a *left scope* (its left operand) and a *right scope* (its right operand): the left and right scopes together are said to constitute *the* scope of the operator. In example (10) the left scope of the $\vee$ is q, and its right scope is $\sim p$; so here p lies in the scope of two $\sim$s and one $\vee$. In example (11) the p lies in the left scope of one $\supset$, but in (12) the p lies in the left scope of two $\supset$s.

$$p \supset (q \supset r) \tag{11}$$

$$(p \supset q) \supset r \tag{12}$$

Because of our top-down approach, at each stage of translation after the first the component being translated will lie in the scope of PL-operators only (not English operators). Our PL-operators are: $\sim$, &, $\vee$, $\supset$, $\equiv$, $\not\equiv$. If the component lies in the scope of a $\equiv$ or $\not\equiv$ then it can be shown that an equivalent translation is required for it. When a component is translated from English to PL we say that it has been *replaced* by the PL component.

Rule 1: If the component lies in the scope of a $\equiv$ or $\not\equiv$, then to preserve the arrow (↔, → or ←) adopt an equivalent translation.

If the component does not lie in the scope of a $\equiv$ or $\not\equiv$, then a non-equivalent translation may be used. In this case the component will be either "odd" or "even" as described below. Recall that the numbers 1, 3, 5, ... are odd whereas 0, 2, 4, ... are even. If the component lies in the scope of t tildes and in the left scope of h hooks then compute $n = t + h$: if n is odd the component is said to be *odd*; if n is even the component is said to be *even*. Consider the following examples:

$$(A \text{ only if } B) \ \vee \ \sim(A \text{ or } B) \tag{13}$$

$$\sim(B \text{ iff } A) \ \supset \ (A \text{ but not } B) \tag{14}$$

In (13), (*A only if B*) is even since it has $n = 0$, but (*A or B*) is odd since it has $n = 1$ (it lies in the scope of one $\sim$). In (14), (*B iff A*) is even since it has $n = 2$ (it lies in the scope of one $\sim$ and in the left scope of one $\supset$), and (*A but not B*) is even since it has $n = 0$. We now state the following rule (without proof).

Rule 2: If the component does not lie in the scope of a $\equiv$ or $\not\equiv$, then:–

(a) to preserve $\rightarrow$: if the component is *even* replace it with a proposition that *it implies*; if the component is *odd* replace it with a proposition that *implies it.*

(b) to preserve $\leftarrow$: if the component is *odd* replace it with a proposition that *it implies*; if the component is *even* replace it with a proposition that *implies it.*

At last we are in a position to determine whether the translation from (4) to (5), repeated below, preserves the $\rightarrow$.

$$\sim(D \text{ or } E) \supset \sim L \tag{4}$$
$$\sim(D \vee E) \supset \sim L \tag{5}$$

In (4), *(D or E)* does not lie in the scope of a $\equiv$ or $\not\equiv$, but it does lie in the scope of one $\sim$ and in the left scope of one $\supset$: so here $n = 2$ and *(D or E)* is even. To preserve $\rightarrow$, we use Rule 2a: we must replace *(D or E)* with a proposition that *(D or E)* implies. Since *(D or E)* $\rightarrow$ $(D \vee E)$, the translation adopted in (5) *will* preserve the arrow. Since the $\rightarrow$ has been preserved at each step, we know that the English proposition (1) $\rightarrow$ the PL proposition (5).

Let's see how the method works on some more examples. Consider the following proposition.

If whenever she laughs people feel good inside then she is not a bad person. (15)

Using:
L = She laughs
G = People feel good inside
B = She is a bad person

we translate to the following equivalent proposition:

If (whenever L, G) then not B (16)

Translating the main operator gives:

(whenever L, G) $\supset$ not B (17)

Since (15) $\leftrightarrow$ (16) and (16) $\rightarrow$ (17), we have (15) $\rightarrow$ (17) i.e. the initial arrow is $\rightarrow$. Can we preserve this all the way through to the end? Let's see. Since *not B* $\leftrightarrow$ $\sim B$ the arrow is preserved when we go to (18):

(whenever L, G) $\supset \sim B$ (18)

Propositions of the form *whenever p, q* are standardly translated as $p \supset q$. If we do this we obtain:

$$(L \supset G) \supset \sim B \tag{19}$$

Was the $\rightarrow$ preserved here? In (18), *(whenever L, G)* has $n = 1$ since it lies in the scope of one $\sim$ only: so here *(whenever L, G)* is odd. Rule 2a now directs us to replace *(whenever L, G)* with a proposition that implies it. But it is *not* the case that $(L \supset G) \rightarrow$ *(whenever L, G)*; in fact the reverse is true. So (19) fails to preserve the $\rightarrow$: it is not true that (15) $\rightarrow$ (19). In a case like this, about the best we can do is change our dictionary so that the troublesome component is represented by a single letter. Using

W = Whenever she laughs people feel good inside

(15) may now be translated as:

$$W \supset \sim B \tag{20}$$

Though detail has been lost here, at least we have (15) → (20).

As we will see later, it is sometimes important to translate so that the English proposition is implied by the PL proposition. In these cases we need to preserve the ←. Consider the following example:

It's not true that if it rains then it pours. (21)

Given an obvious dictionary, this translates to the equivalent:

not (if R then P) (22)

Since *not p* ↔ $\sim p$, (22) translates to the equivalent:

$\sim$(if R then P) (23)

Since (21) ↔ (23), both → and ← have been preserved up to this point. In (23) the component *(if R then P)* is odd since it lies only in the scope of one $\sim$. Suppose we now translate thus:

$$\sim(R \supset P) \tag{24}$$

Before reading on, use Rule 2 to determine which of → and ← was preserved by this step. Since it is true that *(if R then P)* → $(R \supset P)$, and the component is odd, it follows that the ← is preserved. It is not true however that $(R \supset P)$ → *(if R then P)*; so the → is not preserved.

NOTES

In Staines' terminology, what we have called an "odd component" is a "purely negative occurrence of a component", and our "even component" is a "purely positive occurrence of a component".

EXERCISE 7.4

1. For each of the following propositions:

   (i) semi-symbolize the proposition by symbolizing the main operator only, and set out the initial arrow.
   (ii) If the initial arrow can be preserved using *only* the dictionary provided set out the full translation.
   (iii) If the initial arrow cannot be preserved using *only* the dictionary provided set out the dictionary required and the full translation.

   Let: R = It rains.
   F = There is a flood.
   C = The bridge will collapse.

   (a) If it rains, then if there is a flood the bridge will collapse.
   (b) If it does not rain then there will not be a flood.
   (c) If it rains, then if there is no flood the bridge will not collapse.
   (d) The bridge will not collapse if it's not true that if it rains then there is a flood.
   (e) The bridge will collapse if it's true that if it rains then there is a flood.
   (f) The bridge will collapse only if it's true that if it rains then there is a flood.
   (g) The bridge will collapse only if it's not true that if it rains then there will be a flood.
   (h) If it neither rains nor is there a flood then the bridge will certainly not collapse.

(i) It's not true that if there is a flood only if it rains, then the bridge will not collapse.
(j) It's not true that if there is a flood only if it rains, then the bridge will collapse only if there is a flood.
(k) Either there will be a flood if it rains, or the bridge will not collapse.
(l) Either it's not true that if there is a flood the bridge will collapse, or it's not true that if it rains there will be a flood.

7.5 ADEQUATE TESTS FOR PROPOSITIONS AND RELATIONS

Before using Staines arrows in the analysis of arguments, we investigate their use in determining the modal status of propositions and relations. In earlier chapters, although implied translations were often used in testing modal properties and relations, it was assumed that this would have no serious consequences for the adequacy of the testing procedures to handle the original English propositions. In point of fact, such an assumption is not always justified; and it is now time we looked at this more carefully.

To simplify our discussion we now introduce a convention which will be adopted for the rest of this chapter only. When speaking *generally* about a proposition expressed in English as compared with the proposition expressed by its PL translation we will use a *capital letter* for the *English proposition* and the corresponding *lower case letter* for the *PL proposition.* We will standardly place the English proposition on the left and the PL proposition on the right. For example,

$$P \rightarrow p$$

indicates that the English proposition P implies the PL proposition p.

Suppose we want to test whether some English proposition P is a necessary truth. We begin by translating P to p, and then test whether p is necessary. Let us consider first the case where we are able to show that p is necessary (e.g., by using truth tables or possible-truth tables). Showing that p is necessary will not always guarantee that P is necessary. Consider the following translation for instance, using the dictionary: R = He read a book; F = He fell asleep.

If he read a book and fell asleep, then he fell asleep and read a book. (1)

$$(R \,\&\, F) \supset (F \,\&\, R) \qquad (2)$$

Although (2) is a tautology, and hence a necessary truth, if we take "and" in (1) to be used in a temporal sense, it is clear that (1) is not a necessary truth. What we need then, is some condition which lets us know when the proving of p to be necessary is adequate for proving P necessary. We now use the definition of necessary implication to derive such a condition. If $P \leftarrow p$ then in every possible world that p is true, P will be true too. So if p is necessary, and $P \leftarrow p$, it follows that P is true in all possible worlds i.e. P is necessary. So we have:

If p is a necessary truth, and $P \leftarrow p$, then P is a necessary truth.

In other words, the above result says that only necessary truths are implied by necessary truths. Note the direction of the arrow in this result. The opposite relation $P \rightarrow p$ would not be adequate, since any proposition implies a necessary truth (see §4.5).

In the case of (1) and (2) above, we do not have that (1) ← (2); and so the necessity of (2) does not establish the necessity of (1). Let's apply this adequacy result to a couple more examples. Suppose we translate (3) as (4), using: R = It's raining.

It's raining or it's not raining. (3)

$$R \not\equiv {\sim}R \tag{4}$$

Since we can show by PC that (4) is a necessary truth, and we know that (3) ← (4), this shows that (3) is a necessary truth. Suppose however that we adopted the following translation for (3).

$$R \vee {\sim}R \tag{5}$$

Unless we knew the "or" in (3) was inclusive, showing (5) to be necessary would not immediately establish that (3) was necessary because at this stage we would not know that (3) ← (5). However, we can show indirectly that (3) ← (5) by proving that (4) ↔ (5). Notice that since $p \vee q \leftarrow p \not\equiv q$, it follows that if $p \not\equiv q$ is a necessary truth then $p \vee q$ is too. So in practice, if we want to prove an English disjunction is a necessary truth and we aren't sure which type of disjunction is intended, it is best to translate with $\not\equiv$ rather than $\vee$.

Now consider (6) translated as (7).

If it's raining then it's raining. (6)

$$R \supset R \tag{7}$$

Showing that (7) is necessary will not immediately be adequate for showing that (6) is necessary, since at this stage we do not know that (6) ← (7). One way to proceed here is to use our English intuitions to see that (6) is equivalent to:

Either it's not raining or it is raining. (8)

and then see in English that (8) ↔ (3). Our proof that (3) is necessary now establishes that (6) is necessary.

Now let's look at the case where the PL proposition p is shown *not* to be necessary. This means we have found a possible world in which p is false. Under what condition will this test be adequate to show that the English proposition P is not necessary? Well, if $P \rightarrow p$ then there is no possible world with P true and p false; so if p is false in some possible world then P must be false there too i.e. P is not a necessary truth. So we have:

If p is not a necessary truth, and $P \rightarrow p$, then P is not a necessary truth.

Notice that the arrow here is the reverse of that in the previous result. As an example, consider (9) translated as (10) with an obvious dictionary.

If the sun is shining then it's raining. (9)

$$S \supset R \tag{10}$$

A possible-truth table reveals that (10) is false in the possible world where $S = 1$ and $R = 0$. So (10) is not necessary. Since (9) → (10) this proves that (9) is not necessary.

We now set out adequacy conditions for testing self-contradictions and various modal relations. The proofs of these conditions may be developed quickly from the definitions, and are left as an exercise.

If p is a contradiction, and $P \rightarrow p$, then P is a contradiction.
If p is not a contradiction, and $P \leftarrow p$, then P is not a contradiction.

Relation established	Adequate conditions for same relation to hold between P and Q
p is equivalent to q	$P \leftrightarrow p$, $Q \leftrightarrow q$
p is contradictory to q	$P \leftrightarrow p$, $Q \leftrightarrow q$
p is contrary to q	$P \rightarrow p$, $Q \rightarrow q$
p is subcontrary to q	$P \leftarrow p$, $Q \leftarrow q$
p implies q	$P \rightarrow p$, $Q \leftarrow q$
p is implied by q	$P \leftarrow p$, $Q \rightarrow q$
p is indifferent to q	$P \leftrightarrow p$, $Q \leftrightarrow q$
p is consistent with q	$P \leftarrow p$, $Q \leftarrow q$
p is inconsistent with q	$P \rightarrow p$, $Q \rightarrow q$

If any of these relations is shown *not* to hold between p and q, then the same relation will not hold between P and Q provided the two conditions listed in the relevant row of the above table hold, with all the single arrows *reversed*.

Although we have taken p, q to be PL propositions, and P, Q to be English propositions, the adequacy results in this section (and the next) are quite general for any propositions expressed in any language, natural or artificial or mixed.

EXERCISE 7.5

1. For each of the following insert the appropriate Staines arrow between the English and the PL proposition. Then state whether determining the modal status of the PL proposition will be adequate for determining the modal status of the English proposition.

Dictionary: L = Peter is a law student
P = Peter is a philosophy student
M = Peter is a mathematics student

(a) Peter is a law student and Peter is not a law student. $L \,\&\, \sim L$
(b) Peter is either a law student or not a law student. $L \not\equiv \sim L$
(c) Peter is either a law or a philosophy student, or both. $L \vee \sim P$
(d) If Peter is a mathematics student then he is not a mathematics student. $M \supset \sim M$
(e) If Peter is a law student and a philosophy student then he is a philosophy student and a law student. $(L \,\&\, P) \supset (P \,\&\, L)$
(f) It's obvious to everyone that if Peter is a law student then Peter is either a philosophy or a law student. $L \supset (P \vee L)$

2. If we use only the given dictionary for translating the following pairs of propositions, in which cases will PL be adequate to test for (i) contraries, (ii) implication, and (iii) converse implication? Where PL is adequate, carry out the test.

Dictionary: C = The clock is slow
B = The bus is late
E = The bus is early

(a) P If the clock is slow then the bus is late.
Q Either the clock is not slow or the bus is late or both.
(b) P The bus is not both late and early.
Q The bus is either not late or not early or both.
(c) P The clock is slow and the bus is late.
Q The clock is slow but the bus is not late.

(d) P If the clock is slow then the bus is late.
Q If the clock is slow then the bus is not late.
(e) P Either the bus is early or it is late.
Q If the bus is not late, then it is not early.

7.6 ADEQUATE TESTS FOR ARGUMENTS

Let us refer to an argument whose propositions are expressed in English as an *English argument*, and an argument expressed in PL as a *PL argument.* In this section we examine the conditions under which the validity or invalidity of an English argument follows from that of its PL translation. First we will deal with the case where the PL argument can be proven valid (e.g., by truth tables or possible-truth tables). In setting out examples we will use Staines arrows to display the relation between the English and PL propositions involved.

Let's begin with a very simple example.

$$\begin{array}{lcl} \underline{\text{You are intelligent and kind}} & \leftrightarrow & \underline{I \,\&\, K} \\ \therefore \text{ You are intelligent} & \leftrightarrow & \therefore\ I \end{array}$$

Here there is no problem because the two arguments are equivalent. The PL argument is valid, and so the English argument is valid too (and no doubt sound!). Usually however, the arguments will not be equivalent. Fortunately, we don't need them to be. *So long as each English premise implies the corresponding PL premise, and the PL conclusion implies the English conclusion, if the PL argument is valid then the English argument is valid too.* This general result may be set out schematically as follows.

Adequate Validity Test:

$$\begin{array}{lccc} \text{If} & P_1 & \rightarrow & p_1 \\ & \vdots & & \vdots \\ & \underline{P_n} & \rightarrow & \underline{p_n} \\ & C & \leftarrow & c \\ \\ \text{then} & \mathit{VALID} & \leftarrow & \mathit{VALID} \end{array}$$

This result may be easily proved. Recall that an argument is valid iff the conjunction of its premises necessarily implies the conclusion. Note also that → is *transitive* (i.e. given any propositions p, q, r: if $p \rightarrow q$ and $q \rightarrow r$ then $p \rightarrow r$. This follows obviously from the definition of →. Transitivity will be dealt with in detail in Ch. 16). Now if $P_1 \rightarrow p_1, \ldots, P_n \rightarrow p_n$ then $(P_1 \,\&\, \ldots \,\&\, P_n) \rightarrow (p_1 \,\&\, \ldots \,\&\, p_n)$. If the PL argument is valid then $(p_1 \,\&\, \ldots \,\&\, p_n) \rightarrow c$. So if $c \rightarrow C$ we now have $(P_1 \,\&\, \ldots \,\&\, P_n) \rightarrow C$ i.e. the English argument is valid.

Let's look at some more examples. To simplify things a bit, English arguments will be abbreviated with the aid of propositional constants. Three valid argument-forms considered in §4.6 were AA, DC and DD. As the examples below show, English arguments of these forms can be proven valid by testing their PL translations.

$$\begin{array}{lcl@{\qquad}lcl@{\qquad}lcl} \text{If } A \text{ then } B & \rightarrow & A \supset B & \text{If } A \text{ then } B & \rightarrow & A \supset B & A \text{ or } B & \rightarrow & A \vee B \\ \underline{\quad A \quad} & \rightarrow & \underline{A \quad} & \underline{\text{Not } B} & \leftrightarrow & \underline{\sim B} & \underline{\text{Not } A} & \leftrightarrow & \underline{\sim A} \\ \therefore\ B & \leftrightarrow & \therefore\ B & \therefore \text{ Not } A & \leftrightarrow & \therefore\ \sim A & \therefore\ B & \leftrightarrow & \therefore\ B \end{array}$$

In each of these cases we have the English premises implying the PL premises, and the

PL conclusion implying the English conclusion. The next argument however fails the adequacy test:

	It's false that, you are intelligent only if you're kind	←	$\sim(I \supset K)$
∴	You are intelligent and not kind	↔	$\therefore I \,\&\, \sim K$

The negand in the premise is an odd component, so the arrow points to the left but not to the right (see Rule 2b in §7.5). Although the PL argument is valid, the adequacy test is not passed: so it does not automatically follow that the English argument is valid. This is good news actually. If you look at the English argument, you wouldn't want to call it valid anyway.

Here is one example however where the English argument is obviously valid but the standard PL translation of it will typically be unable to show this.

If *A* then *B*	→	$A \supset B$
If *B* then *C*	→	$B \supset C$
∴ If *A* then *C*	→	$\therefore A \supset C$

The PL argument is valid but, except in those rare cases where *If A then C* is seen in English to be equivalent to *Either not A or C or both* and hence is equivalent to $A \supset C$, the PL conclusion does not imply the English conclusion. So the validity of the PL argument does not automatically guarantee that the English argument is valid. Of course, the English argument is valid: both it and the PL argument are said to have the valid form of "Chain Argument". But the validity of the "if ... then ... -version" of Chain Argument depends on more than the validity of the "⊃-version". The following related argument however can be adequately tested in PL.

	A	↔	A
	If *A* then *B*	→	$A \supset B$
	If *B* then *C*	→	$B \supset C$
∴	*C*	↔	$\therefore C$

As one last example, we note that the English form of Contraposition is not validated by the PL version:

	If *A* then *B*	→	$A \supset B$
∴	If not *B* then not *A*	→	$\therefore \sim B \supset \sim A$

Again, the arrow between the conclusions does not go in the required direction. So the validity of the English form of Contraposition depends on more than the validity of the PL form.

So if we want to use PL to prove the validity of English arguments we need to watch out for cases where the adequacy arrows are not satisfied. In particular, be on the lookout for negated conditionals in the premises, and conclusions that are either conditionals or disjunctions. Disjunctive premises, when not clearly exclusive, should be translated with $\vee$; disjunctive conclusions, when not clearly inclusive, should be translated with $\not\equiv$. In drastic cases, negated conditional premises might be treated as simple negations and conditional conclusions might be treated as atomic propositions.

Finally, let's turn to the case where the PL argument can be shown *invalid* (e.g., by possible-truth tables). Recall that an argument is invalid iff there is a counterexample to it i.e. there is a possible world where all the premises are true and the conclusion is false. We leave the proof of the following result as an easy exercise. *So long as each PL premise implies the corresponding English premise, and the English conclusion implies the PL* *conclusion, if the PL argument is invalid then the English argument is invalid too.* Schematically we have:

Adequate Invalidity Test: If

$$\begin{array}{ccc} P_1 & \leftarrow & p_1 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \underline{P_n} & \leftarrow & \underline{p_n} \\ C & \rightarrow & c \end{array}$$

then *INVALID* ← *INVALID*

Notice that the arrows required for invalidation are simply the reverse of those required for validation.

Consider the following example.

$$\begin{array}{lcl} \underline{\text{Lee smokes}} & \leftrightarrow & \underline{S} \\ \therefore \text{ If Lee smokes then he drinks} & \rightarrow & \therefore\ S \supset D \end{array}$$

The PL argument is invalid, as can be shown by producing the counterexample: $S = 1$, $D = 0$ (i.e. a possible world where Lee smokes but does not drink). Since the required arrows are present, this automatically proves the English argument is invalid. But now consider the following example.

$$\begin{array}{lcl} \text{If Lee smokes then he drinks} & \rightarrow & S \supset D \\ \underline{\text{Lee does drink}} & \leftrightarrow & \underline{D} \\ \therefore \text{ Lee smokes} & \leftrightarrow & \therefore\ S \end{array}$$

A possible-truth table generates the following counterexample for the PL argument: $S = 0$, $D = 1$ (i.e. a possible world where Lee drinks but does not smoke). So the PL argument is invalid; in fact it commits the fallacy of Affirming the Consequent (see §4.6). This does not establish the invalidity of the English argument however, since we do not have the required arrow for the first premise. In cases like this we should take the counterexample to the PL argument (let us call this a *PL counterexample*) and, working in English, try to construct out of it a counterexample to the English argument (an *English counterexample*). In this example, any possible world where Lee drinks but does not smoke will constitute a PL counterexample: there are infinitely many such possible worlds: using our English intuitions we should be able to see that in some (but not all) of these the first English premise will be true e.g., suppose Lee is determined that if ever he does take up smoking he will continue his drinking. We now have an English counterexample and the English argument may be declared invalid.

EXERCISE 7.6

For each of the following arguments, set out a dictionary and then translate it into PL. Then test the PL argument for validity. Then use Staines arrows to determine whether your test was adequate for determining the validity of the English argument.

Note: In each of the arguments, certain letters have been emphasised. These should be used for your dictionary. For example, the dictionary for Question 1 will be: G = Susan has a good excuse; E = Susan is exempted from attendance at practical class; H = Susan stays at home.

1. Only if Susan has a *g*ood excuse is she *e*xempted from attendance at the practical class. If Susan stays at *h*ome then she is exempted from attendance at the practical class. But, since she does not have a good excuse, she will not stay at home.

2. We get *m*oney and *s*taff, if either the Budget is *i*ncreased or we *w*rite a good submission. Since we get no staff, we did not write a good submission.

3. Student *n*umbers will be increased if and only if the Universities Commission increases the *q*uota. Either the Government will not be *k*ind or more money will be given for *e*ducation. So, if the Government is kind, and more money will be given for education only if the Universities Commission increases the quota, then student numbers will be increased.

4. The symbol for tungsten is *W* if and only if it is not *T*. This follows from the following assumptions: either the symbol for tungsten is not T or else wolfram is a *m*etal; either the symbol for tungsten is T and is not W or else wolfram is neither a metal nor a *g*as; if wolfram is not a metal then it is not a gas either; and finally, if the symbol for tungsten is T then the symbol is T and not W.

5. Either the manager will *a*gree or he will *f*ire you. If the project is *g*ood he will agree. He will fire you if the project is *u*neconomical. So the project is either good or uneconomical.

6. If the wheat *g*rew and it did not *r*ain during October, then there would have been a good *h*arvest. But there was not a good harvest, even though the wheat grew. So it must have rained during October.

7. If people are *o*ppressed then there will be *v*iolence. Why? Because, people are not oppressed unless *f*orce is used. If force is used then either there is *c*omplete submission or there is *r*esistance. If there is resistance there will be violence, and there is never complete submission.

8. If Jones becomes *i*ll only if the wattle tree *b*looms, then he is either *a*llergic to wattle or something *e*lse is causing his illness. Since he is not allergic to wattle but he has become ill, it follows that something else is causing his illness.

9. John will *b*e embarrassed only if he *f*eels embarrassed. So, if he does not feel embarrassed he will not be embarrassed.

10. Uranium should be *m*ined only if there is a foolproof way of *d*isposing of the waste products. Since there is no such foolproof method, it follows that it's not the case that uranium should be mined.

11. Weather forecasting is an exact *s*cience. Hence it will either *r*ain or not rain tomorrow.

12. If the Premiers do not get more *m*oney they will cut back on *s*ervices to the public. If they do cut back on such services, then the public at large will *p*rotest. If the Premiers do get more money then the *t*axpayers will protest. So, either the public at large or the taxpayers will protest.

13. Jones will *m*eet the contract deadline only if there has been no *s*trike. Since there has been no strike it follows that he will meet the deadline.

14. Philosophy will be *o*ffered externally only if both the Board of *E*xternal Studies and the *F*aculty approve. So either the Faculty approves or Philosophy will not be offered externally.

15. The Government will continue to *c*ut expenditure if there is an *i*ncrease in the deficit. There is an increase in the deficit only if either tax *r*evenue is decreasing or *s*pending is increasing. But spending is not increasing. Since tax revenue is decreasing, the Government will continue to cut expenditure.

16. If John Dunmore Lang had been *f*riendly with Samuel Marsden then Lang would not have said that Marsden *w*as 'Slimy Sam'. Now, both *L*ang and *M*arsden were clerics, and if they were then they *s*hould have been friendly. Since Lang did say that Marsden was 'Slimy Sam', it follows that although they should have been friendly they were not.

17. *U*nemployment will decrease if, and only if there is an upswing in the *e*conomy. If the Government decreases *s*pending, then only if there is to be *m*assive private investment will there be an upswing in the economy. Since there will not be massive private investment, and since the Government will decrease spending, it follows that unemployment will not decrease.

18. The Church supports *s*elf-determination only if it supports the *d*ecisions of the aboriginal people. If the Church supports the decisions of the aboriginal people and the aboriginal people do not *w*ant a State takeover, then the Church will *o*ppose a takeover. The aboriginal people do not want a State takeover. Hence, if the Church does not oppose a takeover, it does not support self-determination.

19. If the Minister acted *p*rudently and *r*eceived the best available advice, then if it was correct for the law to be *a*ltered without any opposition there would have been no *c*ontroversy. But there has been controversy. Since the Minister received the best available advice, it follows that the Minister did not act prudently.

20. If either *R*od or *H*ink invented the arguments then, if the arguments are either about *p*olitical characters or about *m*oral systems then those arguments are both *v*alid and *s*ound. Only if the arguments are *i*nstructive is it untrue that the arguments are both *t*rivial and sound. Thus, either the arguments are instructive or it is false both that Rod invented those arguments and that the arguments are about political characters.

21. If the government *w*anted to encourage the people to conserve petrol then it would *a*llow the import of fuel saving cars. Either the government is not *r*eally interested in conserving fuel or it would want to encourage the people to conserve petrol. Since the government does not allow the import of fuel saving cars, it follows that it is not really interested in conserving fuel.

22. If I am *e*arly at the post office there will be no *q*ueue. If there is no queue I will waste time waiting around *a*fter being served. If I am *l*ate at the post office there will be a queue. I will waste time waiting around *b*efore being served if there is a queue. If I am early I'll not be late. So, either I'll waste time waiting around after being served or before being served.

23. If the *v*acuum cleaner and the *s*tove are both switched on, then the *p*ower goes off. Since the stove is not switched on, the power will not go off.

24. The theory that spacemen carved the Easter Island statues is *p*lausible only if the locals could not have *c*arved the statues themselves. The locals could have carved the statues themselves if there is a *f*ilm of the locals carving a statue in recent times. Since there is such a film, it follows that the theory about spacemen carving the statues is not plausible.

25. Either we don't really *b*elieve in our basic principles or we will not *s*upport the executive, because the executive is *p*assing resolutions which are contrary to our basic principles. And, if we really believe in our basic principles, we must *o*ppose resolutions which are contrary to our basic principles. Furthermore, if we support the executive and the executive is passing resolutions which are contrary to our basic principles, then we will not be opposing resolutions contrary to our basic principles.

26. Either we will *p*lan ahead for adequate water resources or we will not have *e*nough water. We will plan ahead for adequate water resources only if we *a*llocate funds for research. So, if we do not allocate funds for research we will not have enough water.

27. The education system will be able to *h*elp students to reason and think if there are *l*ogic courses in the schools. If the education system is able so to help students, then it is on the way to being a *g*ood system. If the education system is not able so to help students, then it is not on the way to being a good system. So, either the education system is on the way to being a good system or there are no logic courses in the schools.

28. If Jones' theory were *c*orrect then there would have been a *d*isaster in 1975. But since there was no such disaster, it follows that his theory is not correct.

29. If there is neither *s*trife nor *u*nrest in the factory, then the union and the management are *c*o-operating. But since there is strife it follows that the union and management are not co-operating.

30. Either the statement under discussion is a *d*isjunction or it is an *e*quivalence. If the statement under discussion is an equivalence, then it contains an *i*mplication. The statement under discussion does not contain an implication. Therefore, it is a disjunction.

31. The lights will *t*urn on only if the fuses are *i*ntact. If the *p*ower gets to the house then the fuses are intact. So, if power gets to the house the lights will turn on.

32. If terrorists' *d*emands are met, then threats of *v*iolence will be rewarded. If terrorists' demands are not met, then innocent *h*ostages will be murdered. So, either threats of violence will be rewarded or innocent hostages will be murdered.

33. If rain *c*ontinues, then the river *r*ises. If rain continues and the river rises, then the bridge will be *w*ashed away. If rain continues only if the bridge washes away, then a *s*ingle road is not adequate for the town. Either a single road is adequate or the traffic engineers have made a *m*istake. Therefore, the traffic engineers have made a mistake.

34. If Mary goes to *l*ive in a College and gives in to her inclinations, then her social life will *f*lourish. If her social life flourishes then her academic record will *s*uffer. Mary will give in to her inclinations but her academic record will not suffer. So, we

conclude that she will not go to live in a College.

35. If James gets a *s*alary rise or *i*nherits a large sum of money, then he will *b*uy a new sports car. James will *r*ace in the rally only if he buys a new sports car. So it follows that James will race in the rally only if he either gets a salary rise or inherits a large sum of money.

36. Since Kevin was not *e*lected, it follows that either most *p*references went to his chief opponent or his campaign was not on *k*ey issues. If most preferences did not go to his chief opponent then Kevin would have been elected.

37. Either *J*oh will give in or *M*al will. Joh will not give in. If Mal gives in then he will *t*ell a good story. If he tells a good story his party will *b*ow out of the argument. So, Mal's party will bow out of the argument.

38. Either the *p*epper or the *s*inging makes the baby sneeze. If the pepper makes the baby sneeze then Alice should take him *o*utside. If the singing makes him sneeze then either Alice *w*ill prevent the singing or she should take him outside. Since Alice will not prevent the singing, it follows that she should take the baby outside.

39. If the consumption of petrol is not *c*ut then the city atmosphere will become more *p*olluted. If the city atmosphere becomes more polluted then more people will go to *l*ive in the country. If more people go to live in the country then more of the *r*ural environment will be exploited. So it follows that if no more of the rural environment is to be exploited, the consumption of petrol will be cut.

40. I *o*rdered turf for the terrace only if I *w*rote out a cheque for \$100.00. If I ordered turf for the terrace it is to *a*rrive next Tuesday. Since I wrote out a cheque for \$100.00, it follows that the turf will arrive next Tuesday.

41. If B. Eagle is planning to be a *c*andidate, then if the reporters had *a*sked him to declare himself then he would *r*efuse to do so. Mr. Eagle is planning to be a candidate, but he has not refused to declare himself. Therefore, the reporters did not ask him to declare himself.

42. Either that figure is a *s*quare or it does not have *f*our sides. That figure is not a square. So, it does not have four sides.

43. If the Department is *r*unning out of money then it will try to *c*ut costs. If it tries to cut costs then the new social science programmes will be *s*helved. The Department is running out of money. So, the new social science programmes will be shelved.

44. Either the maid is *g*uilty or the butler is *l*ying. If the maid is guilty, poor old Lady Maud was *m*urdered for the ruby brooch. Now, the butler was not lying. So, poor old Lady Maud was murdered for the ruby brooch.

45. Either expenditure on education will *in*crease or it will *d*ecrease. If it increases, then *t*axes will increase. If it decreases, then *s*tandards will fall. If standards fall, then *m*oney will be spent and *q*ualified people will be imported from overseas. If either taxes increase or money is spent, then you will *p*ay more. So, you will pay more.

46. Considering that a laboratory will be *b*uilt only if there is a large sum of money *a*vailable and a suitable set of plans are *d*rawn up, and also that a large sum of

money is available and suitable plans have been drawn up, it follows that a laboratory will be built.

47. Either logic *a*pplies to the real world or it is just a *g*ame with marks on paper. If logic is a game with marks on paper, then it is no *u*se. If logic is no use then it should not be *s*tudied for an examination. So if logic should be studied for an examination then it applies to the real world.

48. If every country *s*igned the treaty, then every country would have to *a*llow inspection of its territory. If every country allowed inspection of its territory, it would no longer be true that every country was in *c*omplete control of its territory. If every country was in complete control of its territory, there would be no *w*ars. It follows that if every country signed the treaty then it is false that there would be no wars.

49. John will *w*in only if either James *f*orfeits or Jenny *s*its on the sidelines. So, *either* John will win only if James forfeits, *or* if Jenny sits on the sidelines John will win.

50. There will be a *g*ood season if and only if the cattle *t*hrive, but there won't be a good season. So if it *r*ains then freight rates will go *d*own, since unless the cattle thrive it won't rain.

51. If you have a TV *s*et then you need a *l*icence, and you don't need a licence provided that you *h*ire your set. Of course if you hire a TV set then you have a TV set. If you hire your set then you have to make regular rental *p*ayments, and so it is clear that whenever you have a TV set you have to make regular rental payments.

52. If abolishing the advertising of tobacco *r*esulted in people smoking less then we could *j*ustify abolishing such advertising. Since abolishing the advertising of tobacco does not result in people smoking less we cannot justify abolishing such advertising.

53. Neither *S*ocrates nor *A*ristotle disbelieved in the existence of the external world, and if Socrates did not disbelieve in the existence of the external world, then Plato's dialogues *m*isrepresent the thinking of Socrates. Now either Plato's dialogues do not misrepresent the thinking of Socrates or Aristotle was right in *c*riticising Plato. Evidently, then, Aristotle did not disbelieve in the existence of the external world and was right in criticising Plato.

54. The green post is either *l*onger or *s*horter than the red post. If it is shorter than the red post, it *i*s not the same length as the red post, if it is longer than the red post it is not the same length. Therefore, the green post is not the same length as the red post.

55. The *f*looding of Lake Pedder is justified only if it is not the only *w*ilderness area in Tasmania. Since it is not the only wilderness area in Tasmania the flooding of the Lake is justified.

56. If you *d*ress the timber you must *s*harpen the planer knives; and you must *t*ake down the planer heads if you sharpen the knives. Yet, although you sharpened the knives you did not dress the timber. So it follows that you do not take down the planer heads unless it is not the case that you sharpen the planer knives when you dress the timber.

57. Brown was to tell them where the *m*oney was or he would have been *t*ortured. Since he did tell them where the money was it follows that he was not tortured.

58. The majority of people are at first *c*onfused by logic if there are large numbers of *d*ifferent symbols. The majority of logic students find the subject *u*seful even though the majority of people are at first confused by the subject. If, and only if, there were not large numbers of symbols would the majority of logic students find the subject useful. So, whether there are large numbers of symbols or not, the majority of people are confused at first by logic.

59. Only if we are not *s*uccessful will the operation *f*ail, and we will not be successful. So the operation will fail.

60. Action will be *t*aken only if we can assume that there is a *c*risis. We can assume that there is a crisis if there is enough *e*vidence. Since there is not enough evidence, no action will be taken.

61. James is not both a *d*octor and a den*t*ist. Since he is not a dentist, it follows that he is a doctor.

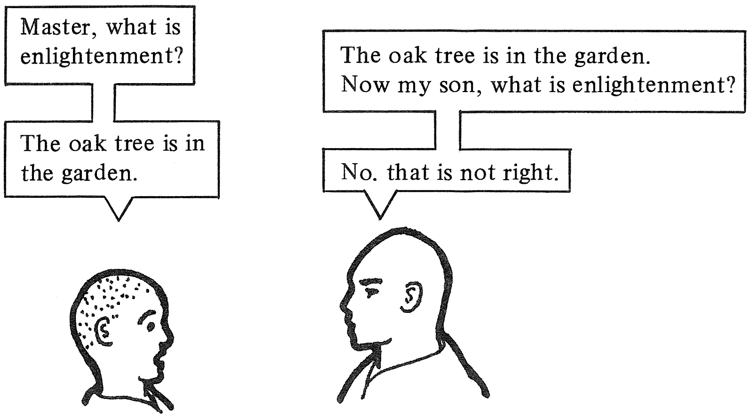
62. If Susan studies *l*ogic then she will *f*ind it useful for all sorts of things. If she studies *c*omputing then she should get a good *j*ob. So, if she studies both logic and computing, she should get a good job.

63. There used to be *c*apital punishment, and at the time when there was the crime rate was very high. If the crime rate was very *h*igh at the time when there was capital punishment then capital punishment was no *d*eterrent against crime. So it is just false that when there used to be capital punishment, it was a deterrent against crime.

64. We will not *a*void water restrictions unless there is *r*ain within six week. There will be no rain within six weeks because Cyclone Bill has *f*orecast a drought. So we will not avoid water restrictions.

65. If the water is *c*lear and *g*erm-free then the purification plant is working. So if the water is not clear the purification plant is not working.

Puzzle 7 Here is a fairly typical dialogue between a zen master and his student. Has the master contradicted himself? Explain.



7.7 SUMMARY

Realistic analysis of everyday argument should take note not only of the *logical* aspect (do the premises support the conclusion as claimed?), but also of the *material* aspect (are the premises true?) and *rhetorical* aspects (e.g., what is the proposer trying to do with his words?).

Identifying an argument entails identifying the conclusion, premises and support claim (deductive or inductive). Arguments with tacit (unstated) premises are called *enthymemes*. In practice it may be necessary to ask the proposer to clarify his argument. The best ways to counter an everyday argument are to state a counterexample in English, or to produce a *counterargument* (an obviously logically defective argument with the same relevant form).

A *countermodel* to an argument expressed in PL is an assignment of truth values to the propositional constants which makes the premises true and the conclusion false: if this describes a *possible* world then it provides a *counterexample*. A countermodel is *factual* iff it describes the actual world. Detection of a non-factual countermodel may suggest additional premises to provide a better argument. Remember that valid arguments may have false conclusions if they have a false premise. In general, arguments may be *modified* by altering the premises, the conclusion, or the support claim.

Translation from English to a logical language is most efficient if an *adequate minimum of detail* is represented. If a logical system is inadequate for evaluation purposes it may be augmented by our English logical intuitions; alternatively, a more powerful logical system may be adequate. The *adequacy* of PL for the analysis of propositions, relations and arguments in English may often be decided with the help of *Staines arrows*. We use ↔, →, ← respectively for necessary equivalence, implication and converse implication. In general we should aim for equivalent translations but this will not always be possible. In the following list, "and" is used in a purely conjunctive sense.

not p	$\leftrightarrow$	$\sim p$	p or q or both	$\leftrightarrow$	$p \vee q$
(Conj.) p and q	$\leftrightarrow$	$p \,\&\, q$	p or q but not both	$\leftrightarrow$	$p \not\equiv q$
p or q	$\rightarrow$	$p \vee q$	neither p nor q	$\leftrightarrow$	$\sim(p \vee q)$
p or q	$\leftarrow$	$p \not\equiv q$	neither p nor q	$\leftrightarrow$	$\sim p \,\&\, \sim q$
if p then q	$\rightarrow$	$p \supset q$	not both p and q	$\leftrightarrow$	$\sim(p \,\&\, q)$
p iff q	$\rightarrow$	$p \equiv q$	not p or not q	$\rightarrow$	$\sim p \vee \sim q$

Some other examples are:

(Temporal) p and q	$\rightarrow$	$p \,\&\, q$	obviously p	$\rightarrow$	p
(Conditional) p and q	$\rightarrow$	$p \supset q$	it's possible that p	$\leftarrow$	p
p is incompatible with q	$\rightarrow$	$\sim(p \,\&\, q)$	it's impossible that p	$\rightarrow$	$\sim p$

In translating complex propositions, first substitute the propositional constants and then proceed top-down. Try to preserve arrows as far as possible. Suppose that a component proposition about to be translated lies in the *scope* of t tildes and in the *left-scope* of h hooks: the component is then said to be *even* or *odd* according as $t + h$ is even or odd.

Rule 1: If the component lies in the scope of a $\equiv$ or $\not\equiv$, then to preserve the arrow (↔, → or ←) adopt an equivalent translation.

Rule 2: If the component does not lie in the scope of a $\equiv$ or $\not\equiv$, then:–

(a) to preserve →: if the component is *even* replace it with a proposition that *it implies*;

		if the component is *odd*	replace it with a proposition that *implies it.*
(b)	to preserve ←:	if the component is *odd*	replace it with a proposition that *it implies*;
		if the component is *even*	replace it with a proposition that *implies it.*

In general results an English proposition may be represented by a capital letter, and the PL proposition by a lower case letter. The following adequacy conditions may be applied to the testing of *modal properties.*

Property established	Adequate condition for same property to hold for P
p is a necessary truth	$P \leftarrow p$
p is not a necessary truth	$P \rightarrow p$
p is a contradiction	$P \rightarrow p$
p is not a contradiction	$P \leftarrow p$

A table of adequacy conditions for testing *modal relations* is supplied in §7.5. Adequacy conditions for testing *arguments* are given below.

$$\begin{array}{llcl@{\qquad\qquad}llcl}
\text{If} & P_1 & \rightarrow & p_1 & \text{If} & P_1 & \leftarrow & p_1 \\
 & \vdots & & \vdots & & \vdots & & \vdots \\
 & \underline{P_n} & \rightarrow & \underline{p_n} & & \underline{P_n} & \leftarrow & \underline{p_n} \\
 & C & \leftarrow & c & & C & \rightarrow & c \\
\multicolumn{4}{l}{\text{then } \mathit{VALID} \leftarrow \mathit{VALID}} & \multicolumn{4}{l}{\text{then } \mathit{INVALID} \leftarrow \mathit{INVALID}}
\end{array}$$

Try to translate so as to meet the adequacy conditions for your particular purposes. When adequacy conditions are not met by the PL translation, try using your logical intuitions in English: in the case of invalidity, a PL counterexample may be helpful in locating an English counterexample.

8 Natural Deduction

8.1 INTRODUCTION

In everyday life, as a way of reasoning something out, we often argue in a step-by-step manner from premises to conclusion. Each step usually involves a simple valid argument-form. To illustrate this method of reasoning, consider the following detective problem:

> If the burglar did not come through the door then he came either across the roof or up the wall. If he came across the roof then someone would have seen him. If he came up the wall then he would have used a ladder. No one saw him. He did not use a ladder.
>
> How did the burglar gain entry?

Before reading on, try solving this problem yourself.

In discussing the problem's solution let us choose the following dictionary:

D = The burglar came through the door
R = The burglar came across the roof
W = The burglar came up the wall
S = Someone saw the burglar
L = The burglar used a ladder

The given facts may now be symbolized as the following premises, which we number for reference.

1.	$\sim D \supset (R \vee W)$	Premise
2.	$R \supset S$	Premise
3.	$W \supset L$	Premise
4.	$\sim S$	Premise
5.	$\sim L$	Premise

Since we are now going to quote some valid argument-forms studied earlier, it would be a good idea if you reviewed the lists of tautological equivalences and valid argument-forms given in §3.9 and §4.7, before continuing.

From lines 2 and 4, by the valid argument-form known as "Denying the Consequent" (or "Modus Tollens"), we may infer:

6. $\sim R$

Similarly, from lines 3 and 5, by Denying the Consequent we deduce:

7. $\sim W$

Now since a conjunction is true if both of its conjuncts are true, from lines 6 and 7 we may infer:

8. $\sim R \,\&\, \sim W$

The rule underlying this latest inference is known as "Conjunction" (Conj): it will be formally defined in §8.3. The next step is to use one of De Morgan's Laws on line 8 to deduce:

9. $\sim(R \vee W)$

Now, from lines 1 and 9, Denying the Consequent yields:

10. $\sim\sim D$

Finally, by Double Negation on line 10 we obtain:

11. D

So we may conclude that the burglar came through the door. A truth table will show that the argument with propositions 1 – 5 as premises and proposition 11 as conclusion is valid. Indeed, all of propositions 6 – 11 follow validly from 1 – 5.

Since this technique of using valid argument-forms to deduce our way step-by-step to the conclusion resembles the way in which we usually reason deductively in everyday life, it is called *"natural deduction"*. When employing this method in logic it is customary to provide a justification column to explain how each line was arrived at. We will use "P" as an abbreviation for "Premise". In most cases, the conclusion to be argued for will be determined at the outset: this will then be displayed to the right of the last premise to remind us of what we are aiming at. Inferences are annotated by quoting the lines and rules used. So the above example would be set out as follows:

1.	$\sim D \supset (R \vee W)$	P
2.	$R \supset S$	P
3.	$W \supset L$	P
4.	$\sim S$	P
5.	$\sim L$	P/ $\therefore$ D
6.	$\sim R$	2, 4 DC
7.	$\sim W$	3, 5 DC
8.	$\sim R \,\&\, \sim W$	6, 7 Conj
9.	$\sim(R \vee W)$	8 DeM
10.	$\sim\sim D$	1, 9 DC
11.	D	10 DN

Such a sequence of propositions is known technically as a "deduction". The line numbers and annotations are not, strictly speaking, part of the deduction. Furthermore, there is usually just a limited set of simple valid argument-forms available for justifying the steps in the deduction. The set we have chosen will be detailed in the next two sections. Later in the chapter, the method of "Conditional Proof" will be introduced to augment the basic deduction method.

The PC methods used in previous chapters are examples of what logicians call "*algorithms*" or "*decision procedures*". An algorithm for a class of problems is a method which *always* produces an answer in a *finite* number of steps, purely by *mechanical* application of the method (which must be describable in terms of a finite list of instructions). For example, the truth-table method provides an algorithm for classifying any PL-argument-form as valid or invalid: you simply apply the procedure and, barring carelessness, you can be sure of getting the right answer. In a similar manner, truth trees and MAV provide an algorithm for this type of problem: once you understand the procedure it becomes just a "turn the handle" process; no original "insights" are required. Is natural deduction also an algorithm for validity determination in PC? Decide for yourself before reading on.

It should be fairly obvious that natural deduction is *not* an algorithmic procedure. Steps in a deduction require use of previous lines and valid argument-forms: but the procedure does not tell us which ones to pick; so rather than the steps being purely mechanical, we have to rely on our own logical insight as to how to proceed. Sometimes we will choose wisely; sometimes unwisely. If we manage to produce a deduction from premises to conclusion then this establishes validity. However, regardless of how much effort has been put into it, *failure to produce a deduction does not establish invalidity:* perhaps the argument-form is invalid; but perhaps it is valid and we simply lacked the ingenuity to produce an appropriate deduction.

Since tables, trees and MAV do provide decision procedures for PC and natural deduction doesn't, why study natural deduction at all? Well for starters, some everyday situations (e.g., dialogue) call for a quick, mental evaluation of arguments: here a mind skilled in natural deduction has a distinct advantage. Secondly, there are many problems in mathematics, science and advanced logic for which there are no known algorithms: here natural deduction is often the best available procedure. Thirdly, even when algorithms do exist, natural deduction lends itself more readily to the discovery of various unanticipated and useful conclusions which follow from a body of evidence. Finally, although the natural deduction approach can at times be frustrating (we may not produce a deduction even after a lot of effort), its essentially challenging nature leads to a greater sense of satisfaction when we do get a deduction out than would be obtained with an algorithmic approach.

Because of the comparatively challenging nature of natural deduction, the exercises will be kept fairly simple until various strategies have been discussed in §8.6. A number of exercises in the next chapter will give you the opportunity to develop your natural deduction skills further.

NOTES

The approach to logic via the notion of deduction was given impetus by a seminar held in 1926 by J. Łukasiewicz. Sometime later both G. Gentzen and S. Jaśkowski published papers setting out systems of natural deduction. Deductive approaches to logic are as old as Aristotle, and were used by Frege and Russell, but deduction itself was not seen as primitive or fundamental in the sense in which deduction is seen in Gentzen's work, especially *Untersuchungen über das logische Schliessen* (Investigation into logical Deduction) published in1935.

Once Conditional Proof is allowed (see § 8.4), an algorithm for natural deduction can be specified.

8.2 RULES OF SUBSTITUTION

In the previous section, we saw how to use simple valid argument-forms such as Double Negation and Denying the Consequent to deduce a conclusion from a set of premises. In natural deduction systems, such forms are usually divided into two separate groups, each group being associated with a particular set of rules. Double Negation finds mention within the *Rules of Substitution*, which we discuss in this section. Denying the Consequent is encompassed by the *Rules of Inference*, which will be treated in the next section.

The Rules of Substitution are more powerful than the Rules of Inference in two ways. First, the Rules of Substitution all work in *both directions*. For example, with Double Negation we know that both of the following are valid:

$$\begin{array}{c} p \\ \hline \therefore \sim\sim p \end{array} \qquad \begin{array}{c} \sim\sim p \\ \hline \therefore \quad p \end{array}$$

Secondly, the Rules of Substitution work on *parts* of formulae. For example, all of the following are proper uses of Double Negation:

$$\begin{array}{c} p \,\&\, q \\ \hline \therefore \sim\sim(p \,\&\, q) \end{array} \qquad \begin{array}{c} p \,\&\, q \\ \hline \therefore \sim\sim p \,\&\, q \end{array} \qquad \begin{array}{c} p \,\&\, \sim\sim q \\ \hline \therefore p \,\&\, q \end{array} \qquad \begin{array}{c} p \,\&\, q \\ \hline \therefore p \,\&\, \sim\sim q \end{array}$$

Though deduction systems may be developed independently of the semantic treatment we have adopted in earlier chapters, it will simplify things in this introductory text if we connect the two approaches. In particular, each Rule of Substitution may be regarded as a statement of tautological equivalence, together with the instruction that the tautologically equivalent items may be substituted one for the other at any point in a deduction. In listing the rules we use "p", "q" and "r" to denote any wff or proposition, and use "::" as an abbreviation for "may be substituted for or replaced by". Our Rules of Substitution, together with their names and abbreviations, are as follows:

Double Negation (DN)
$\sim\sim p :: p$

Commutation (Com)
$p \,\&\, q :: q \,\&\, p$
$p \vee q :: q \vee p$

Association (Assoc)
$p \,\&\, (q \,\&\, r) :: (p \,\&\, q) \,\&\, r$
$p \vee (q \vee r) :: (p \vee q) \vee r$

Distribution (Dist)
$p \,\&\, (q \vee r) :: (p \,\&\, q) \vee (p \,\&\, r)$
$p \vee (q \,\&\, r) :: (p \vee q) \,\&\, (p \vee r)$

De Morgan (DeM)
$\sim(p \,\&\, q) :: \sim p \vee \sim q$
$\sim(p \vee q) :: \sim p \,\&\, \sim q$

Contraposition (Contrap)
$p \supset q :: \sim q \supset \sim p$

Export-Import (Exim)
$(p \,\&\, q) \supset r :: p \supset (q \supset r)$

Material Implication (MI)
$p \supset q :: \sim p \vee q$

Material Equivalence (ME)
$p \equiv q :: (p \supset q) \,\&\, (q \supset p)$
$p \equiv q :: (p \,\&\, q) \vee (\sim p \,\&\, \sim q)$

Idempotence (Idem)
$p \,\&\, p :: p$
$p \vee p :: p$

Exclusive Disjunction (ED)
$p \not\equiv q :: \sim(p \equiv q)$

Most of these are familiar. You may care to check the tautological equivalences for the new ones (e.g., Idempotence) with a truth table.

The fact that the Rules of Substitution may be applied to any well formed part of a formula is very important: you will find several instances of this in the example below. To save writing, a rule may be applied more than once in a single step: in such cases the notation "x n" is appended to the quoted rule to indicate that the rule is applied n times. This practice is illustrated in lines 10, 11 and 12 of the following deduction. Although this example would be a difficult one for you to generate yourself at this stage, it is worth your while to go through it thoroughly, checking that you understand each move in the deduction. As has been done earlier, we will occasionally use propositional constants in examples and exercises without supplying a dictionary.

Example: Use the method of natural deduction to prove the following argument is valid: $A \supset (B \supset C)$ / $\therefore B \supset (A \supset C)$

1. $A \supset (B \supset C)$	P / $\therefore B \supset (A \supset C)$
2. $(A \& B) \supset C$	1 Exim
3. $(B \& A) \supset C$	2 Com
4. $\sim(B \& A) \vee C$	3 MI
5. $(\sim B \vee \sim A) \vee C$	4 DeM
6. $\sim B \vee (\sim A \vee C)$	5 Assoc
7. $(\sim B \& \sim B) \vee (\sim A \vee C)$	6 Idem
8. $(\sim A \vee C) \vee (\sim B \& \sim B)$	7 Com
9. $((\sim A \vee C) \vee \sim B) \& ((\sim A \vee C) \vee \sim B)$	8 Dist
10. $(\sim B \vee (\sim A \vee C)) \& (\sim B \vee (\sim A \vee C))$	9 Com x 2
11. $(B \supset (\sim A \vee C)) \& (B \supset (\sim A \vee C))$	10 MI x 2
12. $(B \supset (A \supset C)) \& (B \supset (A \supset C))$	11 MI x 2
13. $B \supset (A \supset C)$	12 Idem

Since only Substitution Rules have been used in the above deduction, and such rules work in both directions, by reversing the steps above we could show that the converse argument $B \supset (A \supset C)$ / $\therefore A \supset (B \supset C)$ is valid. As a matter of interest, the associated equivalence $p \supset (q \supset r) \Leftrightarrow q \supset (p \supset r)$ is called "Permutation".

NOTES

Rules of Substitution are sometimes called "Rules of Replacement".

The term "idempotence" derives from the Latin *idem* (the same) and *potens* (power), indicating that the first "power" of p is equivalent to its "second power" in the form of $p \& p$ or $p \vee p$, and (by recursion) to any higher power.

The names of the Substitution Rules are not standardised. The most common alternative names are as follows: "Transposition" for Contraposition; "Exportation" for Export-Import; "Implication" for Material Implication: "Equivalence" for Material Equivalence; "Tautology" for Idempotence.

EXERCISE 8.2

1. For each of the following deductions, name the Substitution Rule used.

(a) 1. $\sim A \vee B$ P
2. $\sim A \vee \sim\sim B$

(b) 1. $A \& B$ P
2. $\sim\sim(A \& B)$

(c) 1. $(A \vee B) \supset C$ P
2. $\sim C \supset \sim(A \vee B)$

(d) 1. $A \& \sim(C \vee B)$ P
2. $A \& (\sim C \& \sim B)$

(e) 1. $(A \& A) \supset C$ P
2. $A \supset C$

(f) 1. $(A \& B) \vee (A \& C)$ P
2. $A \& (B \vee C)$

(g) 1. $\sim A \vee (B \& C)$ P
2. $(\sim A \vee B) \& (\sim A \vee C)$

(h) 1. $(\sim A \vee B) \& (\sim A \vee C)$ P
2. $(A \supset B) \& (A \supset C)$

2. In the following deduction some of the entries in the justification column are correct, and some are incorrect. Where incorrect, write the correct justification.

1.	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$	P
2.	$(A \supset (B \supset C)) \supset ((\sim A \vee B) \supset (A \supset C))$	1 MI
3.	$(A \supset (B \supset C)) \supset (\sim\sim(\sim A \vee B) \supset (A \supset C))$	2 DN
4.	$\sim(A \supset (B \supset C)) \vee (\sim\sim(\sim A \vee B) \supset (A \supset C))$	3 DeM
5.	$\sim(A \supset (B \supset C)) \vee (\sim(\sim\sim A \,\&\, \sim B) \supset (A \supset C))$	4 MI
6.	$\sim(A \supset (B \supset C)) \vee (\sim\sim(\sim\sim A \,\&\, \sim B) \vee (A \supset C))$	5 DN
7.	$\sim(A \supset (B \supset C)) \vee ((\sim\sim A \,\&\, \sim B) \vee (A \supset C))$	6 DN
8.	$\sim(A \supset (B \supset C)) \vee ((A \,\&\, \sim B) \vee (A \supset C))$	7 DN
9.	$\sim((A \,\&\, B) \supset C) \vee ((A \,\&\, \sim B) \vee (A \supset C))$	8 Contrap
10.	$\sim((A \,\&\, B) \supset C) \vee ((A \vee (A \supset C)) \,\&\, (\sim B \vee (A \supset C))$	9 Assoc
11.	$\sim((A \,\&\, B) \supset C) \vee ((A \vee (\sim A \vee C)) \,\&\, (\sim B \vee (A \supset C))$	10 MI
12.	$\sim((A \,\&\, B) \supset C) \vee ((A \vee (\sim A \vee C)) \,\&\, (B \supset (A \supset C))$	11 MI
13.	$\sim((A \,\&\, B) \supset C) \vee (((A \vee \sim A) \vee C) \,\&\, ((B \supset (A \supset C))$	12 Assoc
14.	$\sim((A \,\&\, B) \supset C) \vee (((A \vee \sim A) \vee C) \,\&\, (((B \,\&\, A) \supset C)$	13 Exim
15.	$\sim((A \,\&\, B) \supset C) \vee (((A \vee \sim A) \vee C) \,\&\, (((A \,\&\, B) \supset C)$	14 Com
16.	$((A \,\&\, B) \supset C) \supset (((A \vee \sim A) \vee C) \,\&\, (((A \,\&\, B) \supset C)$	15 ME

3. Using the Substitution Rules, construct deductions to show that the following arguments are valid.

(a) $A \;/ \therefore \sim\sim(A \,\&\, A)$
(b) $\sim(A \,\&\, \sim B) \;/ \therefore\; \sim B \supset \sim A$
(c) $A \not\equiv \sim B \;/ \therefore\; (A \supset \sim B) \supset \sim(\sim B \supset A)$
(d) $(A \supset A) \vee (B \supset B) \;/ \therefore\; (A \supset B) \vee (B \supset A)$
(e) $(A \supset A) \supset A \;/ \therefore\; A \vee (A \,\&\, \sim A)$
*(f) $(A \vee B) \,\&\, (B \supset B) \;/ \therefore\; (A \supset B) \supset B$

8.3 RULES OF INFERENCE

We now set out the simpler *Rules of Inference.* Although in a general sense the Substitution Rules considered earlier are used for drawing inferences, what we call Rules of Inference in natural deduction are distinguished by working in *one direction only.* In addition, the Inference Rules are to be used on *whole formulae only*, not parts of formulae. Each of the Inference Rules we discuss here corresponds to a valid argument-form: some of these you have met before; you may wish to verify the others by means of a truth table. Our Inference Rules make no mention of $\equiv$ or $\not\equiv$: these operators are dealt with by means of the Substitution Rules. The simpler Inference Rules are now listed, together with their names and abbreviations.

Simplification (Simp)

$$\frac{p \,\&\, q}{\therefore\; p} \qquad \frac{p \,\&\, q}{\therefore\; q}$$

Conjunction (Conj)

$$\frac{\begin{array}{c} p \\ q \end{array}}{\therefore\; p \,\&\, q}$$

Addition (Add)

$$\frac{p}{\therefore\; p \vee q}$$

Denying a Disjunct (DD)

$$\frac{\begin{array}{c} p \vee q \\ \sim p \end{array}}{\therefore\; q} \qquad \frac{\begin{array}{c} p \vee q \\ \sim q \end{array}}{\therefore p}$$

Affirming the Antecedent (AA)

$$\begin{array}{ll} & p \supset q \\ & p \\ \hline \therefore & q \end{array}$$

Denying the Consequent (DC)

$$\begin{array}{ll} & p \supset q \\ & \sim q \\ \hline \therefore & \sim p \end{array}$$

Chain Argument (ChAr)

$$\begin{array}{ll} & p \supset q \\ & q \supset r \\ \hline \therefore & p \supset r \end{array}$$

Complex Constructive Dilemma (CCD)

$$\begin{array}{ll} & p \vee q \\ & p \supset r \\ & q \supset s \\ \hline \therefore & r \vee s \end{array}$$

These rules must be used in strict accordance with the way in which they are set out. For example, at step 3 in (a) we use DD correctly:

(a)	1.	$\sim A \vee B$	P	
	2.	$\sim\sim A$	P	$/ \therefore B$
	3.	B	1, 2 DD	

But in (b) DD is not correctly used at step 3:

(b)	1.	$\sim A \vee B$	P	
	2.	A	P	$/ \therefore B$
	3.	B	1, 2 DD	

We should have first deduced $\sim\sim A$ as in (c)

(c)	1.	$\sim A \vee B$	P	
	2.	A	P	$/ \therefore B$
	3.	$\sim\sim A$	2 DN	
	4.	B	1, 3 DD	

Can you see the difference between deducing B from 1 and 2 in (b) and 1 and 3 in (c)?

Although in the above examples we have allowed only one rule at each step, for purposes of abbreviation we may combine several moves in one step. The justification for the step should then include all the lines and rules used. For example, the two inferences in (c) may be combined as indicated on line 3 below:

(d)	1.	$\sim A \vee B$	P	
	2.	A	P	$/ \therefore B$
	3.	B	1, 2 DN, DD	

As discussed in the previous section, it is also permissible to use the same rule more than once, in the one step. In the next example AA has been used twice to yield line 4.

(e)	1.	$A \supset (B \supset C)$	P	
	2.	A	P	
	3.	B	P	$/ \therefore C$
	4.	C	1, 2, 3 AA x 2	

Deductions may, of course, be set out in a purely formal way using only forms. In this way we can show that an argument-form is valid by deducing its conclusion from the premises. Consider (f) where we show that Simple Constructive Dilemma (SCD) is a valid form:

(f) 1. $p \vee q$ P
2. $p \supset r$ P
3. $q \supset r$ P $/ \therefore r$
4. $r \vee r$ 1, 2, 3 CCD
5. r 4 Idem

NOTES

The names of the basic Rules of Inference vary from text to text. We have adopted descriptive names where possible. Here is a table of names:

This Text	Traditional	Other
Simp	Simplification	& Elimination
Conj	Conjunction	& Introduction
Add	Disjunctive Addition	$\vee$ Introduction
DD	Disjunctive Syllogism	
AA	Modus Ponens	$\supset$ Elimination
DC	Modus Tollens	
Ch Ar	Hypothetical Syllogism	
CCD	CCD	Constructive Dilemma

EXERCISE 8.3

1. For each of the formulae, other than the premises, in the following deductions insert the correct justification.

(a) 1. $((p \& q) \& r) \& s$ P
2. $p \supset t$ P $/ \therefore t$
3. $(p \& q) \& r$
4. $p \& q$
5. p
6. t

(b) 1. $p \& (p \supset q)$ P
2. $q \supset p$ P $/ \therefore p \& \sim\sim q$
3. p
4. $p \supset q$
5. $(q \supset p) \& (p \supset q)$
6. $p \equiv q$
7. q
8. $\sim\sim q$
9. $p \& \sim\sim q$

(c) 1. $(p \supset q) \supset ((q \supset r) \supset (p \supset r))$ P
2. $\sim\sim(p \equiv q)$ P
3. $(q \supset p) \supset (q \supset r)$ P $/ \therefore \sim\sim(p \supset r)$
4. $p \equiv q$
5. $(p \supset q) \& (q \supset p)$
6. $p \supset q$
7. $(q \supset r) \supset (p \supset r)$
8. $q \supset p$
9. $q \supset r$
10. $p \supset r$
11. $\sim\sim(p \supset r)$

(d) 1. $p \equiv (p \vee p)$ P

2. $p \equiv p$ P

3. p P $/ \therefore p$

4. $(p \supset (p \vee p)) \,\&\, ((p \vee p) \supset p)$

5. $p \supset (p \vee p)$

6. $(p \supset p) \,\&\, (p \supset p)$

7. $p \supset p$

8. $p \vee p$

9. $p \supset p$

10. p

(e) 1. $((p \supset q) \vee (r \supset q)) \equiv ((p \,\&\, r) \supset q)$ P

2. $(p \,\&\, r) \supset q$ P

3. $(p \supset q) \supset q$ P

4. $((r \supset q) \supset q) \equiv ((p \supset q) \supset q)$ P $/ \therefore \sim\sim q$

5. $(p \supset q \;.\vee .\; r \supset q) \supset ((p \,\&\, r) \supset q) \;.$
$\&\; .((p \,\&\, r) \supset q) \supset (p \supset q \;.\vee .\; r \supset q)$

6. $((p \,\&\, r) \supset q) \supset (p \supset q \;.\vee .\; r \supset q)$

7. $(p \supset q) \vee (r \supset q)$

8. $((r \supset q) \supset q) \supset ((p \supset q) \supset q) \;.\&.\; ((p \supset q) \supset q)$
$\supset ((r \supset q) \supset q)$

9. $((p \supset q) \supset q) \supset ((r \supset q) \supset q)$

10. $(r \supset q) \supset q$

11. q

12. $\sim\sim q$

2. Construct deductions to show that the following are valid.

(a) $p \supset q,\ q \supset r,\ \sim r \ / \therefore \sim p$

(b) $p,\ p \supset q,\ q \supset r \ / \therefore r$

(c) $p \vee q,\ q \supset r,\ \sim r \ / \therefore p$

(d) $p \supset r,\ q \supset r,\ \sim r \ / \therefore \sim p \,\&\, \sim q$

(e) $p \supset r,\ q \supset r,\ \sim r \ / \therefore \sim p \vee \sim q$

(f) $p \,\&\, q \ / \therefore p \vee q$

(g) $\sim r \vee \sim s,\ p \supset r,\ q \supset s \ / \therefore \sim p \vee \sim q$

(h) $\sim\sim p \equiv q,\ q \ / \therefore p \,\&\, q$

(i) $(p \,\&\, q) \supset r,\ p,\ p \supset q \ / \therefore r$

(j) $(p \vee q) \supset r,\ q \ / \therefore r \vee s$

8.4 CONDITIONAL PROOF

Sometimes in everyday argument a person will say, "Let us assume, for the sake of argument, that ... ". This type of reasoning is illustrated by the following example.

(a) If the burglar did not come through the door then he came either across the roof or up the wall. If he came up the wall then he would have used a ladder. Let us assume, for the sake of argument, that he did not use a ladder. It follows that he did not come up the wall. From that it follows that if he did not come through the door then he came across the roof. So we may conclude that if he did not use a ladder, then if he did not come through the door he came across the roof.

We can use the dictionary

D = The burglar came through the door
R = The burglar came across the roof
W = The burglar came up the wall
L = The burglar used a ladder

and symbolise (a) to get:

(b)	1.	$\sim D \supset (R \vee W)$	P
	2.	$W \supset L$	P
	3.	$\sim L$	Assumption
	4.	$\sim W$	3, 2 DC
	5.	$\sim D \supset (\sim R \supset W)$	1 DN, MI
	6.	$(\sim D \,\&\, \sim R) \supset W$	5 Exim
	7.	$\sim(\sim D \,\&\, \sim R)$	4, 6 DC
	8.	$\sim\sim D \vee \sim\sim R$	7 DeM
	9.	$\sim D \supset R$	8 DN, MI
	10.	$\sim L \supset (\sim D \supset R)$	Since, given the premises, line 3 leads to line 9

Note carefully the last line. *Given* the facts in the premises, it follows that *if* $\sim L$ *then* $\sim D \supset R$. If you feel uneasy about this step, consult the Notes to this section for an explanation as to why such a move is legitimate.

In deductions, an assumption may be introduced at any point. But the assumption must be used to justify some conditional, and then be *discharged*, before the deduction ends. The conditional established with the aid of the assumption will have the assumption as its antecedent, and the previous line of the deduction for its consequent: check this out for lines 10, 3 and 9 in the example above.

Every assumption has a *scope*, consisting of the assumption itself and every later line of the deduction down to the line before the assumption is discharged. All the lines inside the scope are, in a way, an isolated deduction, except that they usually make use of lines before the scope. Inside the scope we are checking out what follows if we add the assumption to the premises. Once that is checked out, and the assumption discharged, no appeal may be made to lines inside the scope of the assumption. The scopes of assumptions are usually displayed by means of *scope lines*. In the justification column, we will use "A" as an abbreviation for "Assumption", and "CP" as an abbreviation for "Conditional Proof"; when CP is used, the scope of the assumption is also quoted. We now rewrite (b) using these conventions.

(b)	1.	$\sim D \supset (R \vee W)$	P
	2.	$W \supset L$	P
	3.	$\sim L$	A
	4.	$\sim W$	3, 2 DC
	5.	$\sim D \supset (\sim R \supset W)$	1 DN, MI
	6.	$(\sim D \,\&\, \sim R) \supset W$	5 Exim
	7.	$\sim(\sim D \,\&\, \sim R)$	4, 6 DC
	8.	$\sim\sim D \vee \sim\sim R$	7 DeM
	9.	$\sim D \supset R$	8 DN, MI
	10.	$\sim L \supset (\sim D \supset R)$	3–9 CP

The assumption in line 3 has lines 3–9 as its scope. The horizontal line indicates that the assumption has been discharged. More than one assumption may be introduced in a deduction. For example:

(c)	1. $\sim D \supset (R \vee W)$	P
	2. $W \supset L$	P
	3. $\sim L$	A
	4. $\sim R$	A
	5. $\sim W$	3, 2 DC
	6. $\sim R \ \& \sim W$	4, 5 Conj
	7. $\sim\sim(\sim R \ \& \sim W)$	6 DN
	8. $\sim(\sim\sim R \vee \sim\sim W)$	7 DeM
	9. $\sim(R \vee W)$	8 DN
	10. $\sim\sim D$	9, 1 DC
	11. D	10 DN
	12. $\sim R \supset D$	4–11 CP
	13. $\sim L \supset (\sim R \supset D)$	3–12 CP

Here one Conditional Proof has been *"nested"* inside another Conditional Proof. Notice that the scope lines did not cross each other, and that the assumptions were discharged in reverse order. It is important to remember that *scope lines must never cross.*

The general procedure of conditional proof may be summarised in terms of two rules of inference, which we now set out both schematically and in words.

	$\vdots$	
$n.$	p	A
	$\vdots$	
$m.$	q	
	$p \supset q$	$n-m$ CP

Assumption (A) Any assumption may be introduced as a line in a deduction provided it is eventually discharged.

Conditional Proof (CP) A proposition or formula of the form $p \supset q$ may be added (as line $m+1$) to a deduction as soon as q has been deduced (as line m) with the aid of assumption p (on line n, where $n \leq m$), provided that:

there are no undischarged assumptions in lines $n-m$ (the *scope* of p); and

this scope must not be used to deduce any lines after $m+1$.

Note that this procedure allows the following deduction:

(d)	1. p	P / $\therefore$ p
	2. p	A
	3. $p \supset p$	2–2, CP
	4. p	1, 3 AA

This is a case where $n = m$ in the CP rule.

Conditional Proof is particularly useful for deducing conditionals. The usual strategy is to introduce the antecedent of the required conditional as an assumption. This technique is employed twice in the below example, where we first assume the antecedent of the conclusion, and then assume the antecedent of the first assumption.

(e)	1.	$p \supset (q \supset r)$	P / ∴ $(p \supset q) \supset (p \supset r)$
	→2.	$p \supset q$	A
	→3.	p	A
	4.	$q \supset r$	1, 3 AA
	5.	q	2, 3 AA
	6.	r	4, 5 AA
	7.	$p \supset r$	3–6 CP
	8.	$(p \supset q) \supset (p \supset r)$	2–7 CP

Besides offering an alternative and often easier way to demonstrate the validity of various arguments, the method of Conditional Proof expands the range of arguments testable by natural deduction. For example, our previous substitution and inference rules are inadequate to test the obviously valid form $p \supset q$ / ∴ $p \supset (p \,\&\, q)$; we leave it as an easy exercise for you to show this is valid using Conditional Proof.

Although it can be shown that our set of rules, augmented by the method of Conditional Proof, is now adequate for testing any PC-valid argument, it will be useful to add *Reductio ad Absurdum* (RAA) as one of our rules. You have already met the general form of RAA in §4.6, and applied it to both truth trees and MAV. The specific version of RAA that we will use in natural deduction is set out schematically and in words as follows:

	⋮	
→*n*.	p	A
	⋮	
m.	$q \,\&\, {\sim}q$	
	${\sim}p$	*n*–*m* RAA

Reductio ad Absurdum (RAA) To deduce any proposition or formula, assume its negation and then show this leads to a contradiction of the form $q \,\&\, {\sim}q$; then discharge the assumption and deduce the proposition or formula.

Using the above schema, it is easy to show that the RAA principle may be derived from the rules we already have (see the Section Notes). Because CP plays the major role in this derivation, RAA is often viewed just as a special version or application of Conditional Proof.

The RAA technique may be applied to an argument or argument-form as a whole, by negating the conclusion and then showing this leads to a contradiction. Here is a simple example:

(f)

1.	p	P
2.	$p \supset q$	P
3.	$q \supset r$	P $/ \therefore r$
4.	$\sim r$	A
5.	$\sim q$	3, 4 DC
6.	$\sim p$	5, 2 DC
7.	$p \,\&\, \sim p$	1, 6 Conj
8.	$\sim\sim r$	4-7 RAA
9.	r	8 DN

The following longer example illustrates the use of RAA on the argument-form already considered in example (e).

(g)

1.	$p \supset (q \supset r)$	P $/ \therefore (p \supset q) \supset (p \supset r)$
2.	$\sim((p \supset q) \supset (p \supset r))$	A
3.	$\sim(\sim(p \supset q) \vee (p \supset r))$	2 MI
4.	$\sim\sim(p \supset q) \,\&\, \sim(p \supset r)$	3 DeM
5.	$p \supset q$	4 Simp, DN
6.	$\sim(p \supset r)$	4 Simp
7.	$\sim(\sim p \vee r)$	6 MI
8.	$\sim\sim p \,\&\, \sim r$	7 DeM
9.	p	8 Simp, DN
10.	q	5, 9 AA
11.	$q \supset r$	1, 9 AA
12.	r	10, 11 AA
13.	$\sim r$	8 Simp
14.	$r \,\&\, \sim r$	12, 13 Conj
15.	$\sim\sim((p \supset q) \supset (p \supset r))$	2-14 RAA
16.	$(p \supset q) \supset (p \supset r)$	15 DN

NOTES

Conditional Proof may be represented as the following argument-form: p, $(p \,\&\, a) \supset c \;/ \therefore a \supset c$. Here p denotes the conjunction of the premises in the deduction, a denotes the assumption introduced, and $a \supset c$ denotes the line deduced by CP. We leave it as an easy exercise for you to demonstrate the validity of this argument-form, by use of tables, trees or MAV (MAV is quickest).

RAA is often called the method of "Indirect Proof". We now sketch how RAA may be derived from the other rules. From the scope $p, \ldots, q \,\&\, \sim q$ deduce $\underline{p \supset (q \,\&\, \sim q)}$ by CP. Then assume q and deduce $q \supset q$ by CP. Convert this to $\sim q \vee q$ by MI, and by use of Com, DN and DeM convert this to $\underline{\sim(q \,\&\, \sim q)}$. Finally, use DC on the underlined results to deduce $\sim p$. Hence RAA. (We leave the fully detailed proof as an exercise).

With the aid of CP, the truth tree algorithm can be converted into an algorithmic procedure for natural deduction. Moreover, this technique , as developed by one of the authors, can also generate countermodels and hence may be used to establish invalidity etc.

EXERCISE 8.4

1. Complete the justification columns for each of the following deductions.

(a)
1. $p \& \sim q$ P
2. $p \supset q$
3. p
4. q
5. $\sim q$
6. $q \& \sim q$

7. $\sim(p \supset q)$

(b)
1. $p \supset q$ P
2. $\sim q$ P
3. p
4. q
5. $q \& \sim q$

6. $\sim p$

(c)
1. $p \supset r$ P
2. $\sim p \supset r$ P
3. p

4. $p \supset p$
5. $\sim p \vee p$
6. r

(d)
1. $p \vee (q \& r)$ P
2. p
3. $p \vee q$
4. $p \vee r$
5. $(p \vee q) \& (p \vee r)$

6. $p \supset ((p \vee q) \& (p \vee r))$
7. $q \& r$
8. q
9. $p \vee q$
10. r
11. $p \vee r$
12. $(p \vee q) \& (p \vee r)$

13. $(q \& r) \supset ((p \vee q) \& (p \vee r))$
14. $(p \vee q) \& (p \vee r)$

(e)
1. $\sim((p \vee q) \& \sim r)$ P
2. $p \vee q$
3. $\sim(p \vee q) \vee \sim\sim r$
4. $(p \vee q) \supset \sim\sim r$
5. $\sim\sim r$
6. $q \vee p$
7. $(q \vee p) \& \sim\sim r$
8. $(q \vee p) \& r$

9. $(p \vee q) \supset ((q \vee p) \& r)$

2. Construct deductions to show that the following are valid. In each case use Conditional Proof by assuming the antecedent of the conclusion.

(a) $p \supset \sim(q \& r),\ q\ /\ \therefore\ p \supset \sim r$
(b) $p \supset r,\ s \supset \sim r\ /\ \therefore\ s \supset \sim p$
(c) $p \supset (q \vee r),\ s \supset \sim q,\ s \supset \sim r\ /\ \therefore\ s \supset \sim p$
(d) $(p \vee q) \supset r\ /\ \therefore\ p \supset r$
(e) $p \supset r,\ q \supset r\ /\ \therefore\ (p \vee q) \supset r$

3. Construct deductions to show that the following are valid. In each case use Reductio Ad Absurdum by assuming the negation of the conclusion.

(a) $p \supset q,\ \sim p \supset q\ /\ \therefore\ q$
(b) $p \supset q,\ p \supset \sim q\ /\ \therefore\ \sim p$
(c) $p\ /\ \therefore\ q \vee \sim q$
(d) $\sim(p \& \sim q),\ \sim p \supset q\ /\ \therefore\ q$
(e) $p \vee (q \& r),\ r \supset \sim p\ /\ \therefore\ q \vee \sim r$

8.5 THEOREMS AND PROOFS

Now that we allow Assumptions into our deductions we have the possibility of a deduction with no premises. For example:

1.	$p \supset (q \supset r)$	A
2.	$p \supset q$	A
3.	p	A
4.	$q \supset r$	3, 1 AA
5.	q	2, 3 AA
6.	r	4, 5 AA
7.	$p \supset r$	3-6 CP
8.	$(p \supset q) \supset (p \supset r)$	2-7 CP
9.	$(p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$	1-8 CP

The final line of such a deduction is called a *theorem* or *Zero Premise Conclusion* (ZPC). Note that *once proved, a ZPC may be inserted into a deduction at any point.* This is because the deduction which proves the ZPC could be inserted at any point in the overall deduction.

We now set out two further examples of proofs of ZPCs. In the first case the main operator is $\equiv$. So, in this case we prove two ZPCs of the forms $\alpha \supset \beta$ and $\beta \supset \alpha$ and then, by Conjunction and Material Equivalence derive the desired ZPC.

Example 1: Prove: $((p \supset q) \supset q) \equiv (p \vee q)$

1.	$(p \supset q) \supset q$	A
2.	$\sim(p \supset q) \vee q$	1 MI
3.	$\sim(\sim p \vee q) \vee q$	2 MI
4.	$(\sim\sim p \,\&\, \sim q) \vee q$	3 DeM
5.	$(p \,\&\, \sim q) \vee q$	4 DN
6.	$q \vee (p \,\&\, \sim q)$	5 Com
7.	$(q \vee p) \,\&\, (q \vee \sim q)$	6 Dist
8.	$q \vee p$	7 Simp
9.	$p \vee q$	8 Com
10.	$((p \supset q) \supset q) \supset (p \vee q)$	1-9 CP
11.	$p \vee q$	A
12.	q	A
13.	$q \supset q$	12-12 CP
14.	$\sim q \vee q$	13 MI
15.	$(p \vee q) \,\&\, (\sim q \vee q)$	11, 14 Conj
16.	$(q \vee p) \,\&\, (q \vee \sim q)$	15 Com x 2
17.	$q \vee (p \,\&\, \sim q)$	16 Dist
18.	$(p \supset q) \supset q$	17 Com, DN, DeM, MI x 2 (reverse of 1-6)
19.	$(p \vee q) \supset ((p \supset q) \supset q)$	11-18 CP
20.	$((p \supset q) \supset q) \equiv (p \vee q)$	10-19 Conj, ME

In the second example we need to prove the equivalent of the ZPC given. The equivalent is $\sim(p \supset q) \supset (q \supset p)$, so we assume its antecedent.

Example 2: Prove: $(p \supset q) \vee (q \supset p)$

1.	$\sim(p \supset q)$	A
2.	$\sim(\sim p \vee q)$	1 MI
3.	$\sim\sim p \,\&\, \sim q$	2 DeM
4.	$\sim q$	3 Simp
5.	$\sim q \vee p$	4 Add
6.	$q \supset p$	5 MI
7.	$\sim(p \supset q) \supset (q \supset p)$	1-6 CP
8.	$(p \supset q) \vee (q \supset p)$	7 MI

The natural deduction procedure may also be used to show that a set of propositions or formulae is inconsistent. We extend the notion of inconsistency to PL-forms as follows: a set of PL-forms is inconsistent iff there is no model in which each form = 1. It was noted in §4.5 that a contradiction does not follow from anything but a contradiction. So if we *deduce a contradiction* of the form $p \,\&\, \sim p$ from a set, taken as premises, then that *set is inconsistent.*

Before ending the section, we lay down some formal definitions and take note of some important properties of our deduction system.

A *deduction* is a sequence of formulae each of which *either* is a premise, a discharged assumption or a ZPC (with no premise inside an assumption's scope) *or* follows from previous formulae by Substitution or Inference Rules. If $A_1, ..., A_n$ is a deduction without premises then it is a *proof of* A_n, and A_n is a *theorem*. If $A_1, .., A_n$ is a deduction with premises $A_1, ..., A_p$ then it is a *proof of* A_n *from the premises*, and $A_1, ..., A_p$ / $\therefore A_n$ is a valid argument.

Both RAA and ZPC insertion are derived rules, in the sense that they can be derived from the basic rules of our system. The basic rules of our system consist of the Substitution Rules of §8.2, and the Inference Rules consisting of those listed in §8.3 together with Conditional Proof. It can be shown that this deduction system is *consistent* with respect to PC: every theorem of the system is a tautology; every argument provable by the system is PC-valid. It can also be shown that our deduction system is *complete* with respect to PC: every tautology is a theorem; every PC-valid argument is provable by the system.

EXERCISE 8.5

1. Construct deductions to show that the following are ZPCs.

   (a) $(p \,\&\, q) \supset p$
   (b) $((p \supset q) \,\&\, (q \supset r)) \supset (p \supset r)$
   (c) $((p \supset r) \,\&\, (q \supset r)) \supset ((p \vee q) \supset r)$
   (d) $(p \supset (q \supset r)) \equiv (q \supset (p \supset r))$
   (e) $(p \equiv (p \,\&\, q)) \supset (p \supset q)$

2. Construct deductions to show that the following are inconsistent sets.

   (a) $\{p \vee q,\ \sim p,\ q \equiv (q \supset p)\}$
   (b) $\{p,\ \sim p \vee q,\ \sim(p \,\&\, q)\}$
   (c) $\{(p \supset \sim p) \,\&\, \sim(p \supset q)\}$
   (d) $\{\sim(\sim p \vee ((p \supset q) \supset q))\}$
   (e) $\{p \equiv q,\ p \supset \sim p,\ \sim q \supset q\}$

8.6 STRATEGIES FOR PROOF

A fair amount of inventiveness is needed if you are to make proper use of natural deduction. It is therefore useful to know about some of the typical strategies for proof. We will look at five typical strategies for proofs of validity.

The first may be called the *Simple Extraction* strategy. In an argument we have to deduce the conclusion from the premises. So we begin by looking to see whether the conclusion or its negation occurs as a well-formed part of one of the premises. If we can see it there then we may try to "extract" it. For example, consider (a).

(a)	1. $(A \vee B) \supset ((D \supset E) \vee C)$	P
	2. $A \vee B$	P
	3. $\sim C$	P / $\therefore D \supset E$

The conclusion occurs in the consequent of 1, so first the consequent

4. $(D \supset E) \vee C$	1, 2 AA

then the conclusion is extracted

5. $D \supset E$	3, 4 DD

In the next example, (b), the negation of the conclusion occurs in the premises. Find it:

(b)	1. $C \supset D$	P
	2. $A \supset (\sim B \supset C)$	P
	3. $A \mathbin{\&} \sim D$	P / $\therefore B$

The negation of the conclusion is in the second premise. We first get the consequent of the second premise by Simplification from 3 and then AA.

4. A	3 Simp
5. $\sim B \supset C$	4, 2 AA
6. $\sim D$	3 Simp
7. $\sim C$	6, 1 DC
8. $\sim\sim B$	7, 5 DC
9. B	8 DN

The second strategy is used where the conclusion is not a well formed part of any premise. This is the *partial deduction* strategy. When the conclusion is a conjunction we deduce each conjunct. When the conclusion is a disjunction we deduce one disjunct or use Constructive Dilemma.

For example, where the conclusion is a conjunction:

(c)	1. A	P
	2. $A \supset B$	P
	3. $C \vee \sim D$	P
	4. D	P / $\therefore C \mathbin{\&} B$

we first deduce C from 3 and 4

5. $\sim\sim D$	4 DN
6. C	3, 5 DD

then we deduce B

7. B	1, 2 AA

then conjoin them

8. $C \mathbin{\&} B$	6, 7 Conj

If the conclusion is a disjunction then we might need to deduce only one disjunct and then use Addition.

(d)	1.	$\sim F \vee G$	P
	2.	$\sim G$	P / $\therefore H \vee \sim F$
	3.	$\sim F$	1, 2 DD
	4.	$\sim F \vee H$	3 Add
	5.	$H \vee \sim F$	4 Com

If the conclusion is a disjunction and we cannot see how to deduce one disjunct alone, consider the possibility of using Constructive Dilemma. For that we need two conditionals and a disjunction.

The second strategy of partial deduction is really a specific case of the general strategy of looking for *intermediate goals.* In (e) is it clear that we can soon deduce the conclusion if we can deduce a conditional to connect the first and third premises. This gives us an intermediate goal: $B \supset C$

(e)	1.	$A \supset B$	P
	2.	$\sim B \vee (C \vee E)$	P
	3.	$C \supset D$	P
	4.	$\sim E \,\&\, F$	P / $\therefore A \supset D$
	5.	$\sim E$	4 Simp
	6.	$(\sim B \vee C) \vee E$	2 Assoc
	7.	$\sim B \vee C$	5, 6 DD
	8.	$B \supset C$	7 MI (Intermediate Goal)
	9.	$A \supset C$	1, 8 Ch Ar
	10.	$A \supset D$	9, 3 Ch Ar

Sometimes an intermediate goal will be simply to get into the picture a part of the conclusion which does not occur in the premises. Consider the following example.

(f)	1.	$p \supset \sim q$	P
	2.	$r \supset q$	P / $\therefore p \supset \sim r$

Notice that $\sim r$ occurs in the conclusion, while r occurs in the second premise. When conditionals are involved, Contraposition is useful for either introducing or eliminating $\sim$. By applying Contrap to line 2 we are able to get $\sim r$ into the picture (our intermediate goal). The rest of the solution is then obvious, as shown below:

3.	$\sim q \supset \sim r$	2 Contrap
4.	$p \supset \sim r$	1, 2 Ch Ar

A fourth, though related, strategy is to *work backwards* from the conclusion, either mentally or on scratch paper. If you arrive at the premises then simply reverse your steps. Sometimes it is helpful to work forward from the premises and backward from the conclusion, and try to meet in the middle. In working backward, AA and DC tend to be very useful. Consider the following example.

(g)	1.	$p \supset q$	P
	2.	$r \,\&\, \sim q$	P
	3.	$\sim p \supset s$	P
	4.	$s \supset t$	P / $\therefore t$

We want t. We could get this from line 4 by AA if we had s. We could get s from 3 by AA if we had $\sim p$. We could get $\sim p$ from 1 by DC if we had $\sim q$. We can get $\sim q$ from 2 by Simp. We now reverse these steps to give the following solution:

5.	$\sim q$	2 Simp
6.	$\sim p$	1, 5 DC
7.	s	3, 6 AA
8.	t	4, 7 AA

Obviously, this strategy overlaps considerably with earlier strategies.

The fifth strategy is used as a last resort, or as a partial strategy in combination with the other techniques. It is known as the *monkeys on typewriters* approach. Though highly improbable, it is conceivable that a monkey randomly hitting the keys of a typewriter might produce a correct proof simply by accident. So if you haven't got a clue as to how to proceed, just randomly apply the rules and hope that either the conclusion, or at least something useful, will pop up. The name "monkeys on typewriters" is a little misleading, as in order to be sure that you are applying any rule correctly your intellect will need to be involved in *pattern recognition.*

While for the chapter exercises you are expected to limit yourself to the deduction system developed, in practical situations any argument-form deduced to be valid may be used as a derived inference rule (cf RAA earlier). Moreover, while our system of natural deduction is independent of the semantic approaches to PC developed in earlier chapters, it is a fact that substitution rules, inference rules and theorems do correspond to tautological equivalences, PC-valid argument-forms and tautologies. So in practical situations the deduction (or proof-theoretic) approach may be combined with the semantic approaches. For instance, any argument-form shown to be PC-valid by a truth table could be used as an inference rule. On the other hand, the rules of our deduction system could be used to augment the tree method (if you look back to §6.6 you will see that something like this has already been done by the introduction of the Resolution Rule: for example the sub-rule $\alpha \supset \beta$ is just a semantic version of AA.)

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EXERCISE 8.6

1. Use natural deduction to prove that each of the following argument-forms is valid.
   (a) $q \supset r,\ r \supset s,\ q\ /\ \therefore s$
   (b) $\sim p \supset r,\ \sim p\ /\ \therefore q \vee r$
   (c) $p \supset q,\ r \supset p,\ q \supset \sim p\ /\ \therefore r \supset \sim p$
   (d) $q \supset p,\ \sim r \supset \sim p,\ q\ /\ \therefore r \,\&\, \sim\sim p$
   (e) $\sim p \supset q,\ \sim q \,\&\, (r \supset t), p \supset r\ /\ \therefore t \vee t$
   (f) $q \,\&\, r,\ s \supset \sim q,\ p \supset s\ /\ \therefore \sim p$
   (g) $\sim(p \vee q), s \supset p,\ \sim s \supset r\ /\ \therefore r$
   (h) $p \supset q,\ q \supset r,\ \sim r,\ \sim p \supset s\ /\ \therefore s \vee t$
   (i) $r,\ r \supset \sim q, p \supset q,\ s \supset p\ /\ \therefore \sim s$
   (j) $\sim p \,\&\, \sim q,\ r \supset (p \vee q),\ \sim r \supset s\ /\ \therefore t \vee s$
   (k) $p \supset \sim q, \sim p \supset (r \,\&\, s), \sim r \vee \sim s\ /\ \therefore \sim q$
   (l) $p \supset (q \,\&\, \sim r)\ /\ \therefore p \supset (r \supset q)$
   (m) $p \supset (q \supset r),\ r \supset (s \,\&\, t)\ /\ \therefore p \supset (q \supset s)$
   (n) $p \vee (q \,\&\, r), p \supset r\ /\ \therefore r$
   (o) $s \supset (p \supset q), p \supset (q \supset r)\ /\ \therefore s \supset (p \supset r)$
   (p) $[(p \,\&\, q) \supset r] \,\&\, [\sim s \supset (q \,\&\, \sim r)]\ /\ \therefore p \supset s$
   (q) $p \supset \sim p\ /\ \therefore \sim p$
   (r) $q \supset \sim q, (p \,\&\, \sim q) \supset r,\ \sim r,\ (q \,\&\, s) \supset p\ /\ \therefore \sim s \vee \sim q$

2. Provide proofs that the following are ZPCs.
   - (a) $p \vee \sim p$
   - (b) $(p \supset q) \supset ((\sim p \supset q) \supset q)$
   - (c) $p \vee (p \supset q)$
   - (d) $(p \supset q) \supset ((r \vee p) \supset (q \vee r))$
   - $*$(e) $(p \equiv q) \vee (q \equiv r) \vee (p \equiv r)$

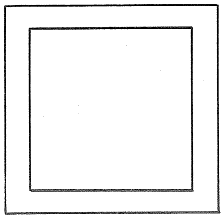
3. Symbolize the following arguments using the emphasized letters. Set your dictionary out. Construct deductions to show that the symbolized arguments are valid.
   - (a) The new logic course will be *o*ffered in 1988 if the Committee *a*pproves. Furthermore, this new course will be offered in 1988 only if some member of staff is *w*illing to teach it. So it follows that if no member of staff is willing to teach this new course the Committee will not approve it.
   - (b) John's political career is at an *e*nd unless his union *b*acks him, because he will *s*tand for election only if his union backs him and if he doesn't stand his political career will be at an end.
   - (c) If Canberra has a *b*each then it is either *n*atural or *a*rtificial. If Canberra's beach is natural then Lake Burley Griffin is a natural *l*ake. The latter is so only if there is no *d*am on the Molonglo River. But since there is a dam on the Molonglo River and Canberra does have a beach, it follows that the beach is artificial.
   - (d) People want somewhere to *l*ive, but if typical suburban homes are *b*uilt then land must be *c*leared. If land must be cleared then we must *r*aze natural bushland. But, since it is utterly false that we must raze natural bushland, it must be concluded that it is also false that if people want somewhere to live then the land must be cleared.
   - (e) My son is *w*atching the Jerry Lewis film if it is *o*n television now. If my son is watching the Jerry Lewis film then the television set will be *t*urned up loud. I *h*ear the set from my study if and only if it is turned up loud. Since I do not hear the set from my study, it follows that the Jerry Lewis film is not now on television.
   - (f) Either studying part-time is *e*asy, or else, if it is not easy, then part-time students either work very *h*ard and do *w*ell or they do not do well. But part-time students do well, and studying part-time is not easy. So it follows that part-time students work very hard.
   - (g) If the extraordinary set is a *m*ember of itself then it is not a member of itself. Since it is a member of itself it follows that it does not *e*xist.
   - (h) The majority of viewers *w*atch light programmes. If there is a real *c*hoice and the majority watch light programmes then the majority *p*refers light programmes. If the majority prefers light programmes but *s*ays that it prefers heavy programmes, then either the majority of viewers are *d*eceiving themselves or they are *l*ying. Since they are not lying and yet say that they prefer heavy programmes, it follows that either the majority of people are deceiving themselves or that there is no real choice.
   - (i) Either all knowledge comes from *e*xperience or all knowledge comes from *r*eason. If all knowledge comes from experience, then logic is *b*ased on experience but not *o*n reason. But logic is based on reason. So, all knowledge comes from reason.

(j) Either taxes will be ***i***ncreased or spending on education will be ***c***ut further. If spending on education is cut further, then people will not be properly ***p***repared for entry to the workforce. Either people will be properly prepared for entry to the workforce or there will be ***e***conomic chaos. But, fortunately, there will not be economic chaos. So, unfortunately, it follows that taxes will be increased.

(k) If I had paid both the ***p***rincipal and the ***i***nterest, I would have ***r***eceived a letter stating that my account is in order. But, if I did not pay the principal or did not pay the interest, then they have probably issued a ***w***arrant for my arrest. I have not received a letter stating that my account is in order. Hence, they have probably issued a warrant for my arrest.

4. If you feel like some more deductions, try to produce a deduction for those arguments shown to be PC-valid in earlier chapters (especially the arguments at the end of Chapter 7).

Puzzle 8



After all those "linear deduction" problems, here is a "lateral thinking" puzzle to stimulate your creativity in a different way.

A fair maiden is trapped on a square island surrounded by a crocodile-infested moat of width 2.4 metres. You are on the outside bank with two planks each of length 2.3 metres.

Suggest two different ways to rescue the fair maiden, one of which depends on her being free to move on the island, and one of which will work even if she is tied up.

8.7 SUMMARY

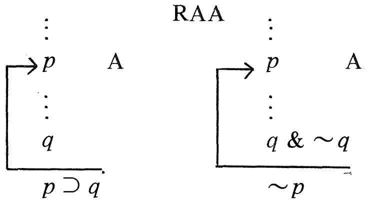
An *algorithm* or *decision procedure* for a class of problems always produces the answer in a finite number of steps purely by mechanical application of its rules. In contrast to tables, trees, and MAV, *natural deduction* is a non-algorithmic procedure for testing PC-validity.* Given our system of rules, an argument is PC-valid iff there is a deduction from premises to conclusion: failure to produce such a deduction however does not prove invalidity.

The *Substitution Rules* correspond to tautological equivalences: they work in both directions and on parts of formulae. The *Inference Rules* correspond to PC-valid argument-forms: they work in one direction only and on whole formulae only. Steps in a deduction should be justified by quoting the rules and previous lines used.

---

*With the aid of CP, the deduction method can be made algorithmic but this formal technique is not treated in this text.

Substitution Rules			
DN	$\sim\sim p$	::	p
Com	$p \& q$	::	$q \& p$
	$p \vee q$	::	$q \vee p$
Assoc	$p \& (q \& r)$	::	$(p \& q) \& r$
	$p \vee (q \vee r)$	::	$(p \vee q) \vee r$
Dist	$p \& (q \vee r)$	::	$(p \& q) \vee (p \& r)$
	$p \vee (q \& r)$	::	$(p \vee q) \& (p \vee r)$
DeM	$\sim(p \& q)$	::	$\sim p \vee \sim q$
	$\sim(p \vee q)$	::	$\sim p \& \sim q$
Contrap	$p \supset q$	::	$\sim q \supset \sim p$
Exim	$(p \& q) \supset r$	::	$p \supset (q \supset r)$
MI	$p \supset q$	::	$\sim p \vee q$
ME	$p \equiv q$	::	$(p \supset q) \& (q \supset p)$
	$p \equiv q$	::	$(p \& q) \vee (\sim p \& \sim q)$
Idem	$p \& p$	::	p
	$p \vee p$	::	p
ED	$p \not\equiv q$	::	$\sim(p \equiv q)$

Inference Rules	
Simp	$p \& q \ / \therefore p$
	$p \& q \ / \therefore q$
Conj	$p,\ q \ / \therefore p \& q$
Add	$p \ / \therefore p \vee q$
DD	$p \vee q, \sim p \ / \therefore q$
	$p \vee q, \sim q \ / \therefore p$
AA	$p \supset q,\ p \ / \therefore q$
DC	$p \supset q,\ \sim q \ / \therefore \sim p$
Ch Ar	$p \supset q, q \supset r \ / \therefore p \supset r$
CCD	$p \vee q, p \supset r, q \supset s \ / \therefore r \vee s$
CP	

Another useful Inference Rule is: SCD $p \vee q,\ p \supset r,\ q \supset r \ / \therefore r$

With *Conditional Proof* (CP) and its derived version *Reductio ad Absurdum* (RAA), any *assumption* (A) must eventually be *discharged*: once an assumption has been discharged, its *scope* must not be used for later inferences. An assumption's scope may be one or more lines long. Scope lines must not cross.

A deduction from a set of formulae to a contradiction of the form $p \& \sim p$ shows the set is *inconsistent*.

A deduction with no premises constitutes a *proof* for the formula on its final line, which is called a *theorem* or *Zero Premise Conclusion* (ZPC). In practice, any known ZPC may be inserted as a line of a deduction. Our system of natural deduction is consistent and complete with respect to PC: its theorems exactly match the tautologies.

Strategies for proof include the *simple extraction* of the conclusion from the premises, deduction of conclusion *parts* for later joining, use of *intermediate goals, working backwards* from the conclusion, and random application of pattern recognition (*monkeys on typewriters*).

9

Further Applications And Notations

9.1 INTRODUCTION

In case you're not already convinced about how marvellously versatile and useful propositional logic is, this chapter should dispel your doubts. In it we investigate further applications, additional operators and alternative notations of propositional logic. To begin with, you will be treated to a modest feast of entertaining logical puzzles (Who said logic wasn't fun?) and shown systematic techniques to help you solve them. After that, we go formal for a while and introduce you, axiomatically, to Boolean Algebra as an uninterpreted calculus. You may be in for a surprise or two when we look at ways of interpreting this algebra. An amazing link up between PC, Set Theory and Switching Calculus will be discovered and put to good use. Our brief look at electronic logic circuits will introduce two more propositional operators and explain what logic gates are all about. We end with a smorgasboard of operators and alternative notations for writing formulae in PC: among other things, this will help you to read those logic books which employ a different notation from ours.

9.2 PUZZLES

Many logicial puzzles begin by listing a set of conditions and then asking you to deduce which option, out of many starters, is the one that is compatible with these conditions. Effectively, they are asking you to produce a possible world which satisfies the conditions of the puzzle: let us speak of such a world as being *satisfyingly possible.* Such puzzles become easier to solve once we have a systematic way of recording the information and of keeping track of which options have been eliminated at the various stages of play. In this connection there are several different types of logic diagrams which may be of use, but we will confine our attention here primarily to *truth tables* and *recording grids.* Logic diagrams are explored further in Chapter 13.

Consider the following problem.

> On being asked what his favourite colour was, John responded as follows:

> It's not red.
> It's blue.
> It's neither red nor blue.
> It's not blue.
>
> A lie detector revealed that John had made just one true statement. What is his favourite colour?

Before looking at the solution below, try your own hand at solving the puzzle. Then write down clearly how you managed to solve it or, if you didn't solve it, what deductions you managed to make before throwing in the towel. (Regardless of whether you solve a puzzle, watching how you think as you try to solve it and working on a clear explanation of your moves are educationally very valuable activities.)

The above puzzle may be solved with a little bit of initiative and some trial-and-error "if ... then ..." reasoning. But it may also be solved quickly and systematically by means of the following truth table. Using the dictionary

R = John's favourite colour is red
B = John's favourite colour is blue

John's four statements are evaluated as shown.

	R	B	$\sim R$	B	$\sim(R \vee B)$	$\sim B$
×	1	1	0	1	0	0
	1	0	0	0	0	1
×	0	1	1	1	0	0
×	0	0	1	0	1	1

Once the standard truth table has been constructed we cross off any rows which are not satisfyingly possible. Here row 1 gets eliminated because its model $R = B = 1$ is simply impossible. This leaves rows 2, 3 and 4 as our (logically) possible-truth table. Now rows 3 and 4 may be eliminated because they fail to satisfy the condition that just one of John's statements is true. The only satisfyingly possible worlds are thus described by row 2, and this has the model $R=1, B=0$. So John's favourite colour is red.

Here's a slightly harder puzzle, where truth tables can help. Try it yourself before looking at our solution.

> During an investigation into the mysterious disappearance of a Mr. Lickit ice-cream van the following statements were made by the prime suspects.
>
> Alan: I wouldn't steal ice-cream unless Charlie helped me.
> Bill: Me? Steal ice-cream? Of course not! I'm too honest for that. Besides, I hate the stuff.
> Charlie: If I pinched it then either Alan or Bill were in it too.
> Des: Neither Charlie nor I were involved.
>
> Given that exactly one of the four suspects is lying, and that exactly two of them were involved in the theft, determine who stole the ice-cream and who is lying.

To solve this, we begin by setting up the following dictionary.

A = Alan stole the ice-cream
B = Bill stole the ice-cream
C = Charlie stole the ice-cream
D = Des stole the ice-cream

Next we symbolize the four claims made by the suspects.

$\sim A \vee (A \,\&\, C)$
$\sim B$
$C \supset (A \vee B)$
$\sim(C \vee D)$

Note that we did not bother to try to symbolize the irrelevant part of Bill's reply. The first claim may also be symbolized as $\sim A \vee C$ since this is logically equivalent to the translation above. While at this point a 16-row truth table may be constructed, it will save us work if we cut down on the number of rows by immediately applying the condition that exactly two of the suspects were involved in the theft. The only options which satisfy this condition are: $A \,\&\, B$; $A \,\&\, C$; $A \,\&\, D$; $B \,\&\, C$; $B \,\&\, D$; $C \,\&\, D$. (Notice the systematic way in which these options were listed: first we list all the pairs containing A, then the remaining pairs containing B, and so on). This yields the following table.

	A	B	C	D	$\sim A$	$\vee$	$(A \,\&\, C)$	$\sim B$	$C \supset$	$(A \vee B)$	$\sim(C$	$\vee D)$
×	1	1	0	0	0	0	0	0	1	1	1	0
	1	0	1	0	0	1	1	1	1	1	0	1
×	1	0	0	1	0	0	0	1	1	1	0	1
×	0	1	1	0	1	1	0	0	1	1	0	1
×	0	1	0	1	1	1	0	0	1	1	0	1
×	0	0	1	1	1	1	0	1	0	0	0	1
						↑		↑	↑		↑	

To satisfy the condition that exactly one of the four suspects is lying, three of the claims should be true and one should be false. This is the case on the second row only. Reading off the information contained in it we conclude: Alan and Charlie stole the ice-cream; Des was lying.

When using truth tables for puzzle analysis, be on the lookout for shortcuts. For example, if all the symbolized propositions are required to be true, as soon as one 0 appears on a row that row may be eliminated. If you know that just one dictionary proposition is true, then reduce the table matrix accordingly; for example, if in the above problem you had been told instead that exactly one of the suspects was involved in the theft you could have got away with just the following matrix:

A	B	C	D
1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1

When puzzle options can be expressed in terms of a small number of propositions, truth tables are usually quite useful. But with some puzzles the options are more efficiently catered for by means of recording grids. The following problem will be used to illustrate the use of such grids; since it is pretty easy to solve anyway, even without the aid of a grid, you might like to have a go at it yourself first before checking our solution.

> Of Alice, Bill and Cathy, one is a logic teacher, one is a science teacher and one is an artist, though not necessarily in that order. The following facts are known:
>
> 1. The two teachers teach at the same school.
> 2. Bill and Cathy live in cities 1000 km apart.
> 3. Alice doesn't know the difference between "if" and "only if".
> 4. Bill is the artist's uncle.

What is each person's occupation?

Figure 1 shows the recording grid at the start of the solution. To save writing the following dictionary has been adopted: *a* = Alice, *b* = Bill, *c* = Cathy, *L* = Logic teacher, *S* = science teacher, *A* = artist. The grid contains nine cells, each of which is empty to begin with. Results deduced as the solution proceeds are entered in the cells according to the following convention: entering a 1 in a cell indicates a matching between the coordinates (i.e. the row and column labels) of the cell; entering a 0 in a cell indicates that the coordinates of that cell do not match. For example, a 1 in the top left cell means that Alice is the logic teacher; a 0 in this cell means that Alice is not the logic teacher. Let's have a look now at the solution. To explain how the grid is filled in we have drawn it more than once so that you can see some of the intermediate stages; in practice all of the entries would be made on the one grid.

	L	*S*	*A*
a			
b			
c			

Fig. 1

	L	*S*	*A*
a			0
b			
c			

Fig. 2

	L	*S*	*A*
a	0		0
b			
c			

Fig. 3

Most puzzles require the solver to make reasonable assumptions. It is reasonable to assume that if the two teachers teach at the same school they do not live in cities 1000 km apart. So from facts 1 and 2 of the puzzle we deduce that Bill and Cathy are not both teachers. Hence Alice is a teacher, and consequently not the artist. This allows us to enter 0 as shown in Figure 2. From fact 3 of the puzzle it is reasonable to assume that Alice is not the logic teacher, so we enter another 0 in the grid as shown in Figure 3. Since each of *a, b* and *c* is to be uniquely matched with one of *L, S* and *A* it follows that Alice is the science teacher. This is recorded as a 1 in the top row of Figure 4. Note that because of the one-to-one matching of coordinates, once a 1 has been entered in a cell all other cells on the same row or column may be assigned 0. So we may enter 0's in the second column (see Figure 4). Now fact 4 (Bill is the artist's uncle) implies that Bill is not the artist: this is recorded as a 0 in the rightmost cell of row 2 as shown in Figure 5. Since, as noted before, each row and each column must have just one cell with the entry 1, all the other values may now be filled in: first we enter 1 in the leftmost cell of row 2, then a 0 in the bottom left cell, and finally a 1 in the bottom right cell (see Figure 6). Our answer is thus: Alice is the science teacher; Bill is the logic teacher; and Cathy is the artist.

	L	*S*	*A*
a	0	1	0
b		0	
c		0	

Fig. 4

	L	*S*	*A*
a	0	1	0
b		0	0
c		0	

Fig. 5

	L	*S*	*A*
a	0	1	0
b	1	0	0
c	0	0	1

Fig. 6

EXERCISE 9.2

1. Mrs Q. T. Pie doesn't always tell the truth about her age. Over the past month she made the following statements:

1. I'm either thirty or forty.
2. I'm not forty.
3. If I'm thirty then I'm not forty.
4. I'm forty.
5. I'm thirty.

What can you deduce about Mrs Pie's age for each of the following cases?

(a) Exactly two of her statements are true.
(b) Exactly three of her statements are true.
(c) Exactly four of her statements are true.

2. Major Colorado is distraught over the theft of his recipe for fried chicken. There are three suspects: Mr. Avarice, Ms Belcher and Ms Crafty. Inspector Hemlock deduces from footprints that exactly two people are involved in the theft. Further investigations reveal the following facts:

    1. Mr. Avarice was involved only if Ms Crafty was not.
    2. Ms Crafty and Ms Belcher are sworn enemies, and would never join each other in any venture.

    Translate conditions 1 and 2 into PL using the following dictionary: A = Mr. Avarice was involved in the theft; B = Ms Belcher was involved in the theft; C = Ms Crafty was involved in the theft. Then use a truth table to determine who the thieves are.

3. Exactly two out of Aaghatha, Boriss, Carveruppa and Draculena murdered Egor and silenced his baying hounds. In his investigations into the crime, Inspector I. D. Duce discovered the following facts:

    1. Boriss was involved only if either Carveruppa or Aaghatha was.
    2. Carveruppa would not take part in the murder if his ghoul-friend Draculena (who has a dental problem) was involved.
    3. Aaghatha murders only on Sundays, and Egor was killed on a Tuesday.

    Can you help Inspector Duce discover the murderers? Provide a dictionary and make use of truth tables.

4. A battle between four exponents of the martial arts is under way.
    The following facts are known.

    1. The good Lord Alpha will not survive if the evil Count Gamma survives.
    2. Either the evil Count Delta will die or the good Lord Beta will die.
    3. Lord Alpha is a master of Kung Fu, while none of the other three are masters; and it is a fact that a master can be beaten only by another master.
    4. Count Gamma will die only if either Lord Alpha dies or Count Delta dies.

    From the above, deduce

    (i) who the certain survivors of the battle are (if any),
    (ii) who the possible survivors of the battle are.

5. In the year 3001, the first galactic beauty contest is held, and female entrants from various planets compete for the position of "Miss Galaxy". Preliminary tests are completed and four entrants are left for the finals:

    Eartha Ekberg from Earth;
    Marilyn Marvellous from Mars;
    Barbra Beautiful from Barnard II; and
    Tina Terrific from Trantor.

These entrants may elect to wear either a swimsuit or an evening gown (but not both) in the final contest.

The following facts are known.

(a) Eartha gets goose pimples when cold, and will wear a swimsuit only if all the others do.
(b) Barbra has a fabulous figure, and is certain to wear a swimsuit.
(c) Marilyn will wear a swimsuit if and only if both Tina and Barbra do.
(d) Tina will not wear a swimsuit if Eartha does.

To minimize any chance of prejudice in the judges, the rules require that more than one person wears a swimsuit.

Determine by means of a truth table who will wear a swimsuit and who will wear an evening gown in the final contest. Provide a dictionary, using the following symbols: *E, M, B, T.*

6. Of Ann, Bill, Cathy and Don one is a Hindu, one is a Christian, one is a Buddhist and one is a Moslem (not necessarily respectively). The following facts are known:

   1. Interested in learning about another religion, both Ann and Bill attended a lecture given by the Hindu.
   2. Either Cathy or Don is a Buddhist.
   3. The Moslem has a long beard.
   4. Don and Ann have never seen each other.

   With the aid of a recording grid, determine each person's religion.

*7. Four men (Alan, Bill, Colin and David) were discussing their wives. They were not well acquainted and the statements they made, as given below, are not all correct. In fact, the only sure thing is that each statement in which a man mentions his own wife's name is correct.

Alan: Karen is Jean's mother.
I have never met Norma.

Bill: Colin's wife is either Karen or Norma.
Jean is the oldest.

Colin: Norma is Alan's wife.
Karen is Jean's older sister.

David: Carmel is my daughter.
Karen is older than my wife.

With the aid of a recording grid, deduce the name of each man's wife.

9.3 BOOLEAN ALGEBRA

The system of Boolean algebra was constructed by the English mathematician George Boole (1815 – 1864), who is regarded by some as the father of pure mathematics. While there are many ways of defining a Boolean algebra, we have chosen to use the set of postulates first enunciated by E. V. Huntington in 1904. These postulates consist of a few axioms (starting assumptions) from which the other theorems of Boolean algebra may be proved. In this section we outline a number of such formal proofs, and draw some contrasts between Boolean algebra and the "ordinary" algebra of school mathematics. The next two sections will consider interpretations and applications of Boolean algebra.

Although set theory will be treated in some detail later, we need to be familiar with a few set notations and concepts at this stage in order to understand what the Boolean axioms actually say. As you probably know, capital letters are usually used to name sets, and when the elements or members of a set are listed, braces are used as delimiters. So if we wanted to use the name "A" for the set which contains just the numbers 1, 2 and 3 we could indicate this by writing "$A = \{1,2,3\}$". The order in which the elements of a set are listed is irrelevant e.g., $\{1,2,3\} = \{2,3,1\}$. The symbol "$\in$" (epsilon) is used as an abbreviation for "is an element of" or "belongs to " e.g., $2 \in \{1,2,3\}$. We also use the symbol "$\notin$" for "does not belong to" e.g., $4 \notin \{1,2,3\}$. Just as p, q etc. are used as propositional variables, we use x, y etc. as element variables when we want to speak generally about elements of sets.

We now define *closure* for sets as follows. If $*$ is a binary operator on elements of a set S, we say that S is *closed under* $*$ iff, given any elements x and y which belong to S, $x * y$ will also belong to S i.e.

S is closed under $*$ iff $x * y \in S$ for all $x, y \in S$

Consider for example the set of natural numbers i.e. $N = \{1,2,3, \ldots\}$. This set is closed under addition since the result of adding any pair of natural numbers will also be a natural number i.e. N is closed under $+$ since $x + y \in N$ for all $x, y \in N$. The term "closure" indicates that if you begin inside the set you can't get outside the set by means of that operator alone i.e. you are closed inside. The set of natural numbers is not closed under subtraction, because by use of subtraction we are able to generate a number outside the set. For instance, $1 \in N$ and $3 \in N$, but $1 - 3 = -2$ and $-2 \notin N$; so N is not closed under $-$.

We now introduce the notion of an identity element. Given any set S on which a binary operation $*$ is defined, we say that e is an *identity element for* $*$ iff, given any element x which belongs to S, $x * e$ is identical to x i.e.

e is an identity for $*$ iff $x * e = x$ for all $x \in S$

For example, consider the set of whole numbers i.e. $W = \{0, 1, 2, 3, \ldots\}$. On this set, 0 is an identity element for addition since $x + 0 = x$ for all $x \in W$.

To specify a Boolean system we need, to begin with, a set of elements on which one unary and two binary operations are defined. In our formal definition the set of elements will be denoted by "S", the unary operation by "$'$" (read as "prime") and the two binary operations by "$+$" (read as "cross") and "$\bullet$" (read as "dot"). Such a system may be represented as $<S, +, \bullet, '>$ where we have used angle-brackets instead of the set-braces to indicate that the order of the items is important. While "$+$" "$\bullet$" and "$'$" have several uses as symbols in mathematics and logic we have chosen non-committal names for them to emphasize that for the moment no specific interpretation is to be attached to them.

With those preliminaries out of the way, we may now define a *Boolean system* as follows.

A system $\langle S, +, \bullet, ' \rangle$ where S is a set of elements on which the binary operations + and • and the unary operation $'$ are defined, is *Boolean* iff it satisfies the following five axioms:

A1. S is *closed* under +, • and $'$

A2. + and • are *commutative*

A3. S contains two elements, 0 and 1, which are *identity elements* for + and • respectively

A4. + and • *distribute over each other*

A5. Each element x of S possesses a *complement* x' such that $x + x' = 1$ and $x \bullet x' = 0$

The theory of Boolean systems is known as *Boolean algebra.*

The symbols "0" and "1", used to denote identity elements in the formal definition, are for the moment uninterpreted. The five postulates may be summarized algebraically as follows, with their descriptive names on the right.

Given any elements $x, y, z \in S$,

A1. $x + y, x \bullet y, x' \in S$ (Clos)

A2. $x + y = y + x$ (Com)
$x \bullet y = y \bullet x$

A3. $x + 0 = x$ (Id)
$x \bullet 1 = x$

A4. $x + (y \bullet z) = (x + y) \bullet (x + z)$ (Dist)
$x \bullet (y + z) = (x \bullet y) + (x \bullet z)$

A5. $x + x' = 1$ (Comp)
$x \bullet x' = 0$

From these five axioms all the theorems of Boolean algebra may be deduced. We will derive eight theorems. This will be enough for you to get the general idea of how the algebra may be built up proof-theoretically, and to acquaint you with some proof techniques which have applications in other areas of logic and mathematics. The theorems have been labelled T1 – T8 for convenience only.

T1. *Principle of Duality (Dual)*

The universal substitution of +, •, 0, 1 for •, +, 1, 0 respectively in any Boolean theorem will produce a Boolean theorem.

Proof: First note that when this substitution is performed on the axioms the axiom set remains the same. Now, for any Boolean theorem there is a proof for which the only theorems quoted are axioms. A proof of the dual theorem is now established by performing the substitution throughout the original proof, using the dual axioms.

For example, campare the parallel proofs below for the theorem $x + (y \bullet 1) = y + x$ and the dual theorem $x \bullet (y + 0) = y \bullet x$.

$x + (y \bullet 1) = x + y$	(Id•)	$x \bullet (y + 0) = x \bullet y$	(Id+)
$= y + x$	(Com+)	$= y \bullet x$	(Com•)

In these two proofs we have indicated the specific versions of the Identity and Commutativity axioms used in the justification column by indicating the specific operator involved. The principle of Duality is a very useful labour-saving device: from now on, once we have proved a theorem we may, by application of this principle, immediately infer the dual theorem.

T2. *Left-Identity*

0 and 1 are left-identities i.e. for all $x \in S$ we have $0 + x = x$, $1 \bullet x = x$

$$\begin{aligned} \textbf{Proof: } 0 + x &= x + 0 && \text{(Com)} \\ &= x && \text{(Id)} \\ \therefore\ 1 \bullet x &= x && \text{(Dual)} \end{aligned}$$

Note that we have used the term "identity" to mean a right-identity. As the above proof reveals, so long as the operator is commutative any right-identity will also be a left-identity.

T3. *Distribution from the right*

+ and • distribute over each other from the right i.e. for all $x,y,z \in S$,

$$\begin{aligned} (x \bullet y) + z &= (x + z) \bullet (y + z) \\ (x + y) \bullet z &= (x \bullet z) + (y \bullet z) \end{aligned}$$

$$\begin{aligned} \textbf{Proof: } (x \bullet y) + z &= z + (x \bullet y) && \text{(Com+)} \\ &= (z + x) \bullet (z + y) && \text{(Dist)} \\ &= (x + z) \bullet (y + z) && \text{(Com+)} \end{aligned}$$

And the second version follows by duality.

Note that we have used "distribution" to mean distribution from the left. As the above proof shows, if the operators are commutative left-distributivity implies right-distributivity.

T4. *Left-Complement*

(Right-) complements are also left-complements i.e. given any $x \in S$, $x' + x = 1$, $x' \bullet x = 0$

$$\begin{aligned} \text{Proof: } x' + x &= x + x' && \text{(Com)} \\ &= 1 && \text{(Comp)} \end{aligned}$$

And the second version follows by duality.

Since commutativity ensures that identities, distribution and complements work on both left and right sides, we often use the terms "Id", "Dist" and "Comp" to justify an inference from either side, with Com being assumed where required.

T5. *Unique Identities*

0 and 1 are unique identity elements in S for + and •

Proof: Let $\triangle$ be an identity element in S for +

Then $\triangle = \triangle + 0$ (using 0 as a right-identity for +)

$= 0$ (using $\triangle$ as a left-identity for +)

$\therefore$ 0 is a unique identity in S for +

$\therefore$ 1 is a unique identity in S for • (Dual)

Note that the above theorem permits us for the first time to speak of 0 as *the* identity in S for +, not just *an* identity. Similarly, 1 is *the* identity in S for •.

T6. *Idempotence*

Given any $x \in S$, $\quad x + x = x$

$x \bullet x = x$

$$\begin{aligned} \textbf{Proof: } x + x &= x \bullet 1 + x \bullet 1 && \text{(Id)} \\ &= (x + x) \bullet 1 && \text{(Dist)} \\ &= (x + x) \bullet (x + x') && \text{(Comp)} \\ &= x + (x \bullet x') && \text{(Dist)} \\ &= x + 0 && \text{(Comp)} \\ &= x && \text{(Id)} \\ \therefore\ x \bullet x &= x && \text{(Dual)} \end{aligned}$$

T7. *Unique Complement*

Each element of S has only one complement.

Proof: Let Δ be a complement of x.

$$\begin{aligned} \text{Then } \Delta &= 1 \bullet \Delta && \text{(Id)} \\ &= (x + x') \bullet \Delta && (x' \text{ Comp}) \\ &= (x \bullet \Delta) + (x' \bullet \Delta) && \text{(Dist)} \\ &= 0 + (x' \bullet \Delta) && (\Delta \text{Comp}) \\ &= (x' \bullet x) + (x' \bullet \Delta) && (x' \text{ Comp}) \\ &= x' \bullet (x + \Delta) && \text{(Dist)} \\ &= x' \bullet 1 && (\Delta \text{Comp}) \\ &= x' \end{aligned}$$

$\therefore\ x'$ is unique

We may now speak of x' as *the* complement of x, rather than just *a* complement of x.

T8. *Involution*

Each element of S equals the complement of its complement
i.e. given any $x \in S$, $x'' = x$

$$\begin{aligned} \textbf{Proof: } \quad & x' + x = 1 && \text{(Comp)} \\ & x' \bullet x = 0 && \text{(Comp)} \\ & \therefore\ x \text{ is the complement of } x' && \text{(df Comp, T7)} \end{aligned}$$

Well that's enough theorems for now. Although there is no point in learning off any of the proofs just considered, we hope you picked up a few ideas from them and appreciated some of their beauty. As you will be aware from your work on natural deduction, proof construction is not only a science but an art as well. Once you have sweated over the creation of some logical proofs yourself, you are able to appreciate the artistry behind a well constructed proof. For instance if you look back to the uniqueness proof for T5, the simplicity and efficiency of its reductio ad absurdum approach make the proof not "just a proof" but an "elegant proof".

To prove some system *is* Boolean, we can try to show either that is satisfies the axioms or that it is isomorphic to (i.e. has the same structure as) a known Boolean system. In the next two sections we will establish the Boolean nature of three particular systems.

It is usually much easier to prove that a system is *not* Boolean, because all we need do is show that it fails to satisfy at least one Boolean axiom or theorem. Let's look at a few examples of this. Suppose we let R denote the set of real numbers and ask whether the system $<R, +, \times, ->$ is Boolean, where +, x and – here are the plus, multiply and unary minus of ordinary algebra. Although this system satisfies axioms A1, A2, A3 and half of A4 (using the numbers 0 and 1 as identities), it fails to satisfy distribution of + over ×, and also fails to satisfy axiom A5 (you will be asked to verify this in the Exercise). So it is not Boolean. We could also show the system is not Boolean by finding a derived theorem which it demonstrably fails to satisfy e.g., clearly the + and × of ordinary algebra are not idempotent (see theorem T6).

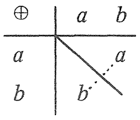
In the above illustration, the set was transfinite (*R* has an infinite number of members) Sometimes we need to test whether or not a *finite* system is Boolean. In such a case it is helpful to use *Cayley tables* for defining the operators (review §2.3 if necessary). Consider for instance the system $< S, \oplus, \odot, '>$ where $S = \{a, b\}$ and $\oplus, \odot, '$ are defined on *S* as follows:

$\oplus$	a	b
a	a	a
b	b	b

$\odot$	a	b
a	b	a
b	a	b

	$'$
a	b
b	c

That this system is not Boolean can be demonstrated in many ways. Three of the easiest tests to conduct at a glance are for *closure, commutativity* and *idempotence. S is closed under an operator iff every element in the operator's table belongs to S.* In the system above, *S* is not closed under $'$ since $b' = c$ and $c \notin S$. A dyadic operator is *commutative iff its table is symmetric about its main diagonal* (i.e. the diagonal from top-left to bottom-right). By "symmetry" here we mean that if you think of the main diagonal as a mirror then each element in the table will match its corresponding "image" by reflection in this mirror. In the table below, the elements joined by dots stand in this object-image relation and yet are not equal. So $\oplus$ is not commutative. Once this diagonal test has located a counterexample to commutativity, this should be specified in full e.g., here we would say:



$$\begin{array}{l} a \oplus b = a \\ b \oplus a = b \\ \quad\;\; a \neq b \end{array}$$

$\therefore \oplus$ does not commute.

A dyadic operator is *idempotent iff the main diagonal of its table matches the table heading.* $\oplus$ is idempotent since both the main diagonal and the heading of its table consist of the sequence $<a,b>$. However $\odot$ is not idempotent, because the top-left element in the diagonal fails to match the coordinate above it.

$\odot$	a	b
a	b	
b		b

This counterexample to idempotence would be specified as follows:

$$\begin{array}{l} a \odot a = b \\ \therefore a \odot a \neq a \end{array}$$

$\therefore \odot$ is not idempotent.

We leave it as an easy exercise for you to show why these diagram inspection tests do provide adequate tests for closure, commutativity and idempotence within a finite system. If a system passes these tests, it could still be non-Boolean: if you can't quickly find another theorem which it fails to satisfy then you should test the other three axioms (A3 – A5); if it passes these as well, it is Boolean.

NOTES

By all accounts, George Boole was not only a brilliant logician but (naturally enough?) a very decent person. For an interesting account of his life see E.T. Bell's *Men of Mathematics* (Penguin, 1953) Ch. 23.

Though we have given the operators + and • equal priority, many authors prefer to give • priority over + and allow concatenation as an abbreviation for •. With those conventions the expression $x + (y \bullet z)$ may be abbreviated to $x + y \bullet z$ or $x + yz$.

EXERCISE 9.3

1. Under what rational operations (+, –, x, ÷) are the following number sets closed?
   - (a) N (Natural Numbers i.e. 1, 2, 3, ...)
   - (b) W (Whole Numbers i.e. 0, 1, 2, 3, ...)
   - (c) Z (Integers i.e. 0, ±1, ±2, ±3, ...)
   - (d) Q (Rationals i.e. x/y where x and y are integers and $y \neq 0$)
   - (e) R (Real numbers)

2. Answer the following with respect to "ordinary algebra" on the real numbers.
   - (a) What are the identity elements for +, –, x, ÷?
   - (b) Which of +, –, x, ÷ are commutative?
   - (c) Does x distribute over +? If not, give a counterexample.
   - (d) Does + distribute over x? If not, give a counterexample.
   - (e) Has each element got a complement (as defined in axiom A5, reading "+", "•" as +, x)? If not, show why not.
   - (f) Is ordinary algebra Boolean i.e. is $<R, +, \times, - >$ a Boolean system?
   - (g) Which of +, –, x, ÷ have left-identities?
   - (h) Does ÷ distribute over + from the right?
   - (i) Does ÷ distribute over + (from the left)?

3. State the dual of the following Boolean theorems.
   - (a) $x + (0 + y) = y + x$
   - (b) $x' + (x \bullet y) = x' + y$
   - (c) $(1 + y)' = 0 \bullet y'$

4. Construct Cayley tables for the following operations on the sets indicated, and use them to investigate properties of ordinary algebra. In each case test for closure, commutativity, idempotence and identities. In the cases of + and x, describe the pattern that emerges for the main diagonal. In the cases of – and ÷, describe the pattern that emerges by considering reflection in the main diagonal.
   - (a) + on the set $\{0, 1, 2\}$
   - (b) – on the set $\{0, 1, 2\}$
   - (c) x on the set $\{1, 2, 3\}$
   - (d) ÷ on the set $\{1, 2, 3\}$

5. Prove the following Boolean theorems.
   - (a) $x \bullet (y + 0) = y \bullet (x + 0)$
   - (b) $x + x + x = x$
   - (c) $x + (y \bullet x) = (x + y) \bullet x$
   - (d) $x'''' = x$
   - (e) $(x \bullet x \bullet x)' + x = 1$
   - (f) $0' = 1$

6. The operations $\#$, $\triangle$ and $'$ on the set $S = \{a, b\}$ are defined by the Cayley tables below. Give three reasons why the system $<S, \#, \triangle, '>$ is not Boolean.

#	*a*	*b*
a	*b*	*a*
b	*a*	*a*

△	*a*	*b*
a	*a*	*a*
b	*b*	*b*

	′
a	*c*
b	*a*

7. Give three reasons why the operators □ and ∇ defined below on the set $S = \{0,1,2\}$ are not Boolean.

□	0	1	2
0	0	1	2
1	1	1	3
2	2	3	2

∇	0	1	2
0	0	1	2
1	1	2	1
2	2	2	2

*8. The system $<S, \oplus, \odot, ' >$ is defined as follows. $S = \{0, 1\}$

⊕	0	1
0	0	1
1	1	1

⊙	0	1
0	0	0
1	0	1

	′
0	1
1	0

   - (a) What are the identity elements for each operator?
   - (b) Prove that the system is Boolean by showing that it satisfies the axioms.

9.4 PC AND SET THEORY

In the previous section Boolean algebra was developed as an uninterpreted calculus. In this section we look at two different realizations of this calculus. Perhaps you can guess what these might be before reading on.

Some of the results in the previous section may have reminded you a bit about propositional calculus. This is understandable because, as it turns out, PC itself is Boolean. Given any non-empty dictionary of propositions, let *S* be the set of propositions in this list together with all elements contructible from this list by means of disjunction, conjunction or negation (Note that *S* must therefore contain a tautology of the form $p \vee \sim p$ and a contradiction of the form $p \,\&\, \sim p$): then $<S, \vee, \&, \sim >$ is a Boolean system. To prove this we need to show that the Boolean axioms are satisfied under this interpretation. The identity sign "=" used in our abstract development will need to be replaced, in the context of PC, with the tautological equivalence symbol "⇔". The abstract identity elements 1 and 0 will be interpreted as T and F respectively, where "T" denotes any tautology and "F" denotes any PC-contradiction. Thus when we speak of T and F as *the* identity elements, their "uniqueness" is qualified by tautological equivalence. Note that the 1 and 0 of abstract Boolean algebra are *not* the truth values 1 and 0 of PC.

Since ∨, & and ∼ are proposition-forming operators the closure axiom is satisfied. The

PC-interpretation of the other four Boolean axioms, together with their PC names, are listed below:

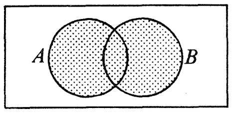
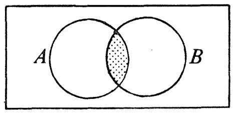
A2.	$p \vee q$	$\Leftrightarrow$	$q \vee p$	(Com∨)
	$p \,\&\, q$	$\Leftrightarrow$	$q \,\&\, p$	(Com&)
A3.	$p \vee \mathsf{F}$	$\Leftrightarrow$	p	(Id∨)
	$p \,\&\, \mathsf{T}$	$\Leftrightarrow$	p	(Id&)
A4.	$p \vee (q \,\&\, r)$	$\Leftrightarrow$	$(p \vee q) \,\&\, (p \vee r)$	(Dist∨&)
	$p \,\&\, (q \vee r)$	$\Leftrightarrow$	$(p \,\&\, q) \vee (p \,\&\, r)$	(Dist&∨)
A5.	$p \vee \sim p$	$\Leftrightarrow$	T	(LEM)
	$p \,\&\, \sim p$	$\Leftrightarrow$	F	(LNC)

Except for Id, these have all been verified before. We leave it as an easy exercise to establish Id by means of a truth table, using the fact that F is always false and T is always true.

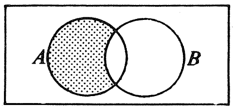
Since PC is Boolean, any abstract Boolean theorem may be interpreted as a tautological equivalence in PC. For instance we now know that Duality holds for tautological equivalences (T1) and that Distribution for &, ∨ works from the right. Idempotence (T6) has been met before in PC under the same name, and Involution (T8) is just Double Negation.

As well as PC, set theory is Boolean. Before establishing this we will deal briefly with a few more basic ideas about sets. Sets, like numbers and propositions, are *abstract* entities, rather than chunks of concrete reality. For any given set there is always a unique answer to the question "How many elements are in the set?". Associated with any chunk of physical reality we can imagine many different sets, depending on what we decide to count as elements e.g., the set of all atoms in this sheet of paper has fewer members than the set of all the elementary particles (electrons, protons etc.) in this sheet of paper. We say that that two sets are *identical* iff they have the same members: this is called the Principle of Extensionality. Sets may be defined by listing their members, or by mentioning an identifying property of their members e.g., "$A = \{x : x$ is a positive integer less than $4\}$" is read "A is the set of all (elements) x such that x is a positive integer less than 4"; here the colon ":" is read "such that". If B is defined by listing as $B = \{1,2,3\}$ then we have $A = B$. The set of all elements in the universe under consideration is called the *universal set* which we denote by $\mathcal{E}$; the set with no members is called the *null set* which we signify by $\{\}$.

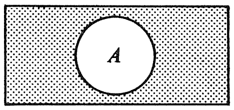
We now define four *operations* on sets, and represent these by means of *Venn diagrams*: here the rectangle stands for the universal set and the shaded area indicates the set being defined.

Operation	*Symbol*	*Definition*	*Diagram*
Union	$A \cup B$	$\{x{:}x \in A \vee x \in B\}$	
Intersection	$A \cap B$	$\{x{:}x \in A \,\&\, x \in B\}$	

Difference $A - B$ $\{x{:}x \in A \;\&\; x \notin B\}$



Complement A' $\mathcal{E} - A$



Examples: If $\mathcal{E} = \{1, 2, 3, ...\}$ $A = \{1, 3\}$ $B = \{2, 3, 4\}$

then $A \cup B = \{1, 2, 3, 4\}$ $A \cap B = \{3\}$ $A - B = \{1\}$

$A' = \{2, 4, 5, ...\}$

We read "$A \cup B$", "$A \cap B$", "$A - B$" and "A'" respectively as "A union B", "A intersect B" "A minus B" and "A complement". We say that A is a *subset* of B iff A has no elements which are not in B. For example, $\{\}$, $\{1\}$, $\{2\}$, $\{1, 2\}$ are the subsets of $\{1, 2\}$. We define the *power-set* of A to be the set of all the subsets of A e.g., the power-set of $\{1, 2\}$ is $\{\{\}, \{1\}, \{2\}, \{1, 2\}\}$.

In providing a set-theoretic interpretation for the Boolean system $<S, +, \bullet, '>$ we replace the formal $+$, $\bullet$ and $'$ respectively with the set-operators $\cup$, $\cap$ and $'$. It will not do to take S as the "set of all sets" as this notion is logically absurd (see supplement for a discussion of Russell's Paradox). S however will be a set of sets, and to ensure that S is closed under $\cup$, $\cap$ and $'$ it will be sufficient if we define S to be the power-set of $\mathcal{E}$, where $\mathcal{E}$ is the given universal set. Check this out for yourself for the case of $\mathcal{E} = \{1, 2\}$. Taking S in this way then, the system $<S, \cup, \cap, '>$ satisfies the closure axiom A1. Before looking at the other axioms let's consider what the identity elements might be. In PC the Boolean identities 1 and 0 were interpreted as T and F, which represent the two extreme types of propositions. Before reading on, see if you can guess what the identity elements will be in set theory.

If you picked the two extreme types of set, you're right: 1 is interpreted as the universal set $\mathcal{E}$ and 0 is interpreted as the null set $\{\}$. The set theory versions of the remaining Boolean axioms may now be set out as follows:

A2. $A \cup B = B \cup A$
$A \cap B = B \cap A$

A3. $A \cup \{\} = A$
$A \cap \mathcal{E} = A$

A4. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

A5. $A \cup A' = \mathcal{E}$
$A \cap A' = \{\}$

Informally, all of these axioms may be swiftly demonstrated by Venn diagrams: the Venn diagram for the left hand expression in each will be seen to match that of the right hand expression. More rigorously, the results may be proved by using the operation definitions given earlier and the corresponding PC results e.g.,

$A \cup B = \{x : x \in A \lor x \in B\}$ (df $\cup$)
$\{x : x \in B \lor x \in A\}$ (Com$\lor$)
$B \cup A$ (df $\cup$)

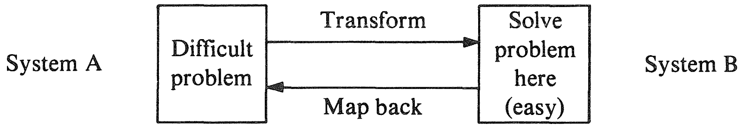
We leave these demonstrations as an exercise for you. In the justification column for the above example we have used "df" as an abbreviation for "definition".

Since both PC and set theory are Boolean in nature, they exhibit the *same formal structure*: we say they are *isomorphic* to each other (from the Greek *isos* = same, *morphus* = shape). The correspondence between the symbols used is summarized in the table below.

Uninterpreted Boolean system	*Propositional Calculus*	*Set Theory*
$x, y, \ldots$	$p, q, \ldots$	$A, B, \ldots$
$+$	$\vee$	$\cup$
$\bullet$	$\&$	$\cap$
$'$	$\sim$	$'$
1	T	$\mathcal{E}$
0	F	$\{\}$
$=$	$\Leftrightarrow$	$=$

As an aid to memorizing the correspondence between the binary operators, note that both "+" and "V" consist of two straight line segments, and that both "V" and "∪" are concave upwards. We have seen that, when given the above interpretations, the Boolean axioms (and consequently all Boolean theorems) are theorems of PC and Set Theory. It can also be shown (we do not prove this in this book) that the Boolean axioms constitute an *adequate* axiom set for PC and Set Theory (i.e. any theorem of PC or Set Theory is deducible from these axioms and is consequently a Boolean theorem). Because of the total correspondence between the uninterpreted system $<S, +, \bullet, '>$, the PC system $<S, \vee, \&, \sim>$ and the set theory system $<S, \cup, \cap, '>$ it follows that *any theorem of one system is also a theorem of the other systems* (when translated into the corresponding symbols).

This isomorphism has useful implications. Firstly it leads to economy, because proof of a theorem in one system proves it also for the other systems. Secondly, it is often easier to "see" and hence remember a particular theorem in one particular system. Thirdly, although parallel techniques may be set up in all the systems, in practice some of these techniques will feel more natural or familiar in one particular system. Hence if we are stuck with a problem that is difficult to handle in one system we may translate or *map* the problem into an isomorphic system, solve it there more easily, and then map the result back into our original system. This general idea is diagrammed below, where system A is taken to be isomorphic to system B.



The first mapping is called a *transformation* and the mapping back is called an *anti-transformation*. People use this technique very often in life, whether they realize it or not e.g., when they apply mathematics to physical reality. As a first approximation, one might regard pure mathematics as the development of abstract systems, physics as the attempt to discover isomorphisms between various aspects of physical reality and abstract systems, and applied mathematics as the practical utilization of the isomorphisms so discovered. We have already used an isomorphism between PC and certain aspects of

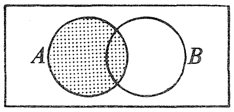
"English Calculus" when we mapped (translated) difficult arguments from English into PC, tested their validity there, and then mapped back the result into English. Within mathematics there are many important examples. One you are probably familiar with is the logarithm transformation: suppose you wish to calculate $(17 \times 23)^8$; this problem is mapped into the "log system" by mapping 17 onto log 17, 23 onto log 23, × onto +, and exponentiation onto ×, to yield $(\log 17 + \log 23) \times 8$; this easier calculation is performed and the anti-log is consulted to map back the solution into our original system. In more advanced mathematics, other types of transformation (e.g., the Laplace transform) can dramatically simplify the solution of difficult problems.

It can be shown that any PC theorem may be represented as a tautological equivalence, and vice versa (see the Notes to this section). Thus when working in PC we may use any of the standard PC techniques (tables, trees, MAV, natural deduction) to establish a Boolean theorem. Because of this, and because of your greater familiarity with PC, we suggest that from now on you use PC to work on any problem that has been formulated in terms of the uninterpreted Boolean system notation. Consider for example, the Boolean theorems listed below as T9 – T13. Their PC versions (and PC names) are listed on the right.

T9.	$1' = 0$	$\sim \mathsf{T} \Leftrightarrow \mathsf{F}$	(NT)
	$0' = 1$	$\sim \mathsf{F} \Leftrightarrow \mathsf{T}$	(NF)
T10.	$x + 1 = 1$	$p \vee \mathsf{T} \Leftrightarrow \mathsf{T}$	(DT)
	$x \bullet 0 = 0$	$p \;\&\; \mathsf{F} \Leftrightarrow \mathsf{F}$	(CF)
T11.	$x + (x \bullet y) = x$	$p \vee (p \;\&\; q) \Leftrightarrow p$	(Abs)
	$x \bullet (x + y) = x$	$p \;\&\; (p \vee q) \Leftrightarrow p$	(Abs)
T12.	$x + (y + z) = (x + y) + z$	$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$	(Assoc)
	$x \bullet (y \bullet z) = (x \bullet y) \bullet z$	$p \;\&\; (q \;\&\; r) \Leftrightarrow (p \;\&\; q) \;\&\; r$	(Assoc)
T13.	$(x + y)' = x' \bullet y'$	$\sim(p \vee q) \Leftrightarrow \sim p \;\&\; \sim q$	(DeM)
	$(x \bullet y)' = x' + y'$	$\sim(p \;\&\; q) \Leftrightarrow \sim p \vee \sim q$	(DeM)

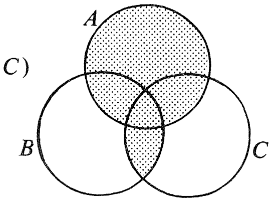
T9 and T10 may be proved swiftly by means of a table, using the fact that a tautology is always true and a contradiction is always false. The labels "NT", "NF", "DT", "CF" are abbreviations for "Negate a Tautology", "Negate a PC-Contradiction", "Disjoin a Tautology" and "Conjoin a PC-Contradiction". As with T9 and T10, theorem T11 (Absorption) is intuitively obvious in its PC version and is easily proved with a table. T12 and T13 have been met before in PC: Associativity is most quickly established by MAV, and DeMorgan's Laws are best handled with tables; the natural deduction proofs for both T12 and T13 are quite lengthy. Note that the theorems have been arranged in dual pairs so that the second member of each pair will follow from the first by Duality.

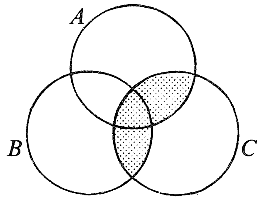
When required to determine whether a proposed set theory equation is a theorem, an informal test may be conducted with Venn diagrams. The equation will be a theorem iff the Venn diagrams for both sides of the equation match in their final shaded areas. Consider for instance the following set version of Absorption: $A \cup (A \cap B) = A$. Here each side of the equation yields the Venn diagram opposite. Alternatively, you could translate the theorem into its PC version, $p \vee (p \;\&\; q) \Leftrightarrow p$, and solve it there. Though Venn diagrams are not considered to provide rigorous proofs, they are useful for finding counterexamples for non-theorems; and once a counterexample has been found it may be



substituted in to provide a rigorous disproof. Consider for instance the non-theorem: $A \cup (B \cap C) = (A \cup B) \cap C$. The Venn diagrams for the left and right expressions are shown below.

$A \cup (B \cap C)$





$(A \cup B) \cap C$

To generate a counterexample here, all we need do is put an element somewhere in the region where the shaded areas differ, and set all the other regions to null. For instance, suppose we put the element 1 in the top region of A and make all the other regions empty. Since the shaded regions represent the two expressions this will ensure that the left expression is the set $\{1\}$ and the right expression is the null set. We verify this counterexample formally by substitution as follows:

$$\text{Let } A = \{1\},\ B = \{\},\ C = \{\}$$

$$\begin{aligned} \text{Then } A \cup (B \cap C) &= \{1\} \cup (\{\} \cap \{\}) &= \{1\} \cup \{\} = \{1\} \\ (A \cup B) \cap C &= (\{1\} \cup \{\}) \cap \{\} &= \{1\} \cap \{\} = \{\} \end{aligned}$$

Though we have made little use of natural deduction in PC for establishing Boolean theorems, this technique is especially useful for *simplification* of Boolean expressions, a matter to be taken up in the next section. To keep your skills up to scratch with this technique we have stipulated in the exercise below that most of the theorems there should be proved by natural deduction. To assist you in this regard you will find a list of relevant theorems in the chapter summary.

NOTES

On set notations, the "$\in$" is the first letter (in Greek) of the word "element", and "$\mathscr{E}$" is the first letter of "Euler" : Leonhard Euler was a famous Swiss mathematician whose diagrams for sets will be discussed in Ch. 13. The null set is frequently represented as ϕ or Λ and the universal set as U or V. The complement A' is often written as $\bar{A}$ or $-A$, and "|" is often used in place of ":" in set definitions.

For ease of application, we have taken Boolean sets for PC to be sets of propositions. In more formal treatments the set of truth values is taken to be the Boolean set for PC i.e. $S = \{1,0\}$. The system defined in Exercise 9.3 Question 8 may then be used to establish the Boolean nature of PC (and later, Switching Calculus) with appropriate substitutions for the operators.

For pragmatic reasons, the term "proof" has been used rather liberally in this section. In the proof-theoretic sense of "proof" as a deduction sequence, truth tables for instance do not provide a proof technique. In practice however such methods do work. In later sections it will be seen that the Boolean operators $\sim$, &, $\vee$ are more than adequate to define the other propositional operators e.g., $p \supset q = \text{df} \sim p \vee q$. Given this, and the use of T and F, it follows that all tautologies may be expressed as tautological equivalences involving no other operators than $\sim$, &, $\vee$ e.g., T: $p \supset p$ becomes $\sim p \vee p \Leftrightarrow$ T. The consistency and completeness of PC (which we do not prove in this book) can now be used to link the tautologies to the theorems of our Boolean PC system.

EXERCISE 9.4

1. Given that $\mathcal{E} = \{2, 4, 6, 8, 10\}$, $A = \{2\}$, $B = \{2, 4, 6\}$, $C = \{4, 8\}$ and $D = \{6, 8, 10\}$ calculate the following.
   (a) $B \cup D$
   (b) $A \cap C$
   (c) C'
   (d) $A \cup (B \cap C)$
   (e) $(B - C)'$

2. The symmetric difference operator $\dot{-}$ is defined as follows:
   $$A \dot{-} B = \{x : x \in A \not\equiv x \in B\}.$$
   Shade in the region $A \dot{-} B$ on a Venn diagram, and by inspection of this diagram state whether the following equality is generally true: $A \dot{-} B = (A \cup B) - (A \cap B)$.

3. Write down the PC and set theory versions of the following Boolean theorems.
   (a) $1 \bullet (x + 0) = x$
   (b) $(x \bullet 1) + (x + y)' = x + y'$

4. Draw Venn diagrams for both the left and right hand expressions in the following equations and state whether or not the equation is a theorem. If it isn't provide a counterexample and verify it by substitution.
   (a) $A' \cup B = A \cap B'$
   (b) $(A \cup B')' = A' \cap (A \cup B)$
   (c) $A \cap (B \cap C) = (A \cup B) \cap (A \cup C)$
   (d) $A \cap B' = [B \cup (A \cap \mathcal{E})']'$

5. Map the formulae of Question 4 onto the corresponding formulae of PC and test them there by any means.

6. Map the following formal Boolean theorems onto PC theorems and prove them by natural deduction in PC. You may use any of the theorems in the Chapter summary.
   (a) $x + (x \bullet x') = x''$
   (b) $(x \bullet 1) + (x \bullet y) = x$
   (c) $x \bullet (y + z)' = (x \bullet y') \bullet z'$
   (d) $[x \bullet (y + z)]' = x' + (y' \bullet z')$
   (e) $x + (x + y)' = x + y'$
   (f) $(x + y) \bullet (x + y') = x$
   (g) $[x + (y \bullet 1)']' = y \bullet x'$
   (h) $(x + y) + [x + (y \bullet z)] = x + y$
   (i) $(x \bullet y) + (y' \bullet x) = x$
   (j) $x' + (x + y)' + [(z + w)' \bullet 0] = x'$
   *(k) $(x + z) \bullet (x' + y) \bullet (y + z) = (x \bullet y) + (x' \bullet z)$
   [Hint for (k): Expand the RHE and try to work your way back to the LHE]

9.5 SWITCHING CIRCUITS AND LOGIC GATES

In this section we investigate some of the applications of PC to electrical and electronic circuits. We begin by looking at Switching Calculus, first developed as a Boolean system by C. E. Shannon in 1938, and make use of PC to simplify and design switching circuits. Then we extend these ideas to the analysis of logic gates.

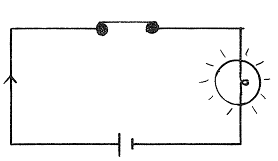
Switching Circuits:

Transmission Values

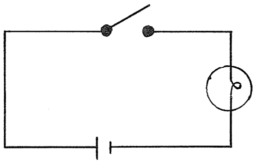
A single switch, or an electrical network in general, is said to *transmit* or to have a *transmission value* of 1 iff a voltage applied across its ends would cause a current to pass through it. If a circuit element will not transmit current it is said to have a transmission value of 0. Various terms are used to describe these two states, e.g.,

1 ... closed, on , conducting
0 ... open , off, non-conducting

If a battery and light are connected in series with a network having a transmission value of 1, the light will go on. If the network has a value of 0, the light will remain off. The simplest network we can consider is that of a single switch, as shown below.



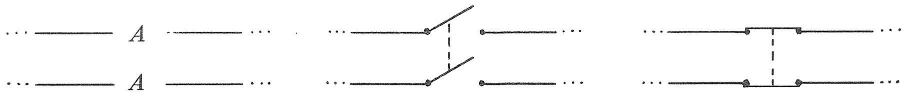
transmission value
of closed switch = 1



transmission value
of open switch = 0

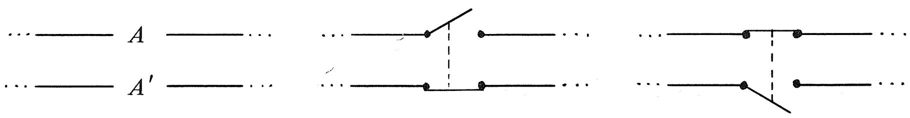
Labels

Many types of switches exist e.g., relays, transistor switches, diode switches. We shall represent all of these in the same way, by capital letter *labels A, B* etc. Switches may be connected non-electrically (e.g., magnetically) so as to always *open and close together*: such switches are given the *same label.* This idea is indicated in the following diagram, where the vertical broken line represents a non-conducting connection between the two switches.



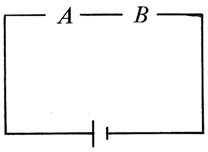
Switches may also be connected so as to always have *opposite* transmission values. This inverse relation is denoted by ′ or ~ . Thus a switch A' (or $\sim A$) is open whenever A is closed and closed whenever A is open. As the transmission table indicates, this corresponds to *negation.* The general idea is indicated in the following diagram.

A	A'
1	0
0	1



Series Connection

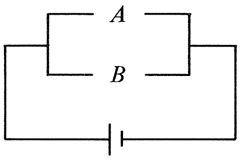
Current can pass through two switches connected in series iff both switches are closed i.e. the A, B series combination has a transmission value of 1 iff both A and B have transmission values of 1. This combination may be symbolized as $A \ \& \ B$ or as $A \bullet B$. As the table indicates this corresponds to *conjunction.*



A	B	$A \ \& \ B$
1	1	1
1	0	0
0	1	0
0	0	0

Parallel Connection

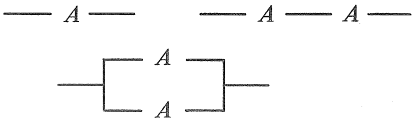
Current can pass through a parallel combination of switches iff at least one of them is closed. This combination may be symbolized as $A \vee B$ or as $A + B$. As the table indicates, this corresponds to (inclusive) *disjunction.*



A	B	$A \vee B$
1	1	1
1	0	1
0	1	1
0	0	0

Equivalence

Two switching networks are said to be *equivalent* iff they have identical transmission values under all conditions i.e. for the same input, each gives the same output. For example, the three networks shown opposite are equivalent: each transmits iff A is closed. We use "⇔" to denote such equivalence, e.g., $A \Leftrightarrow A \ \& \ A, A \Leftrightarrow A \vee A$.

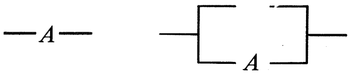


Identity Elements

A network which can always transmit is equivalent to an unbroken wire. Such a network constitutes an *identity element in series*. We will denote such an element by T.

$A \Leftrightarrow A \ \& \ \mathrm{T}$

A network which can never transmit is equivalent to a broken wire. Such a network constitutes an *identity element in parallel.* We will denote such an element by F.



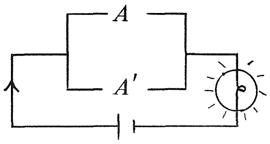
$A \Leftrightarrow A \vee \mathrm{F}$

In terms of switches, T corresponds to a permanently closed switch and F to a permanently open switch.

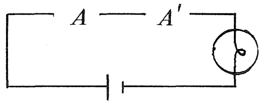
A Boolean System

Let us use the term "network" to describe any single switch (including the special "switches" T and F) or combination of switches which can possibly be formed using the three operations already discussed (viz. opposition, series connection, parallel connection). If we take S as the set of all such networks then S is closed under these operations. Moreover, the correspondence noted earlier between the SC (Switching Calculus) operations and the PC operations of negation, conjunction and disjunction implies that for any network x there corresponds a proposition *"x transmits"* wherein the SC operations $'$, &, $\vee$ may consistently be interpreted as the PC operations $\sim$, &, $\vee$. For instance, the network $A \vee B'$ corresponds to the proposition "the network $A \vee B'$ transmits", which is equivalent to the proposition "A transmits *or* B does *not* transmit". Clearly then, SC is isomorphic to PC, and since PC is Boolean so is SC.

Note that 1, 0, ⇔, T, F of SC correspond to 1, 0, ⇔, T, F of PC. The modal status of PL-forms may thus be tested by constructing the corresponding switching network. Such constructions can be illuminating (in more ways than one!). In the left circuit below, the light stays on in all cases (A closed, A open) thus exemplifying the Law of Excluded Middle. In the right circuit below, the light stays off in all cases, thus exemplifying the Law of Non-Contradiction.



$A \vee \sim A \;\Leftrightarrow\;$ T



$A \;\&\; \sim A \;\Leftrightarrow\;$ F

Simplification of Circuits

Suppose you are given the diagram of a switching network which successfully performs a certain job. You are now required to *simplify* this network as much as possible i.e. you have to produce a diagram of an equivalent network which performs the same job but does so with the minimum possible number of switches. This type of problem has obvious practical applications in the electronics industry where the design of cheaper mass-produced circuits can lead to huge financial savings. The problem may be split up into the following three steps:

1. Write the formula for the network
2. Simplify this as much as possible
3. Draw the diagram for the simpler formula

Sometimes it is easier to see what to do immediately from the diagram, but we shall adopt the following procedure. The formula for the circuit will be written in the notation of PC, and then standard logical methods (especially natural deduction and truth tables) will be used to achieve the simplification.

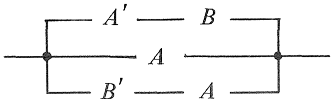
In most cases the quickest method will be natural deduction. Some of the most useful principles in this regard are Absorption, Idempotence, Distribution (in reverse), De Morgan's Laws, and Identity. Idem, Dist and DeM are sometimes used in their n-ary form, as shown below.

$$\begin{aligned}
p \vee p \vee \ldots \vee p \quad &\Leftrightarrow \quad p && \text{(nIdem)}\\
p \,\&\, p \,\&\, \ldots \,\&\, p \quad &\Leftrightarrow \quad p && \text{(nIdem)}\\
p \vee (q_1 \,\&\, \ldots \,\&\, q_n) \quad &\Leftrightarrow \quad (p \vee q_1) \,\&\, \ldots \,\&\, (p \vee q_n) && \text{(nDist)}\\
p \,\&\, (q_1 \vee \ldots \vee q_n) \quad &\Leftrightarrow \quad (p \,\&\, q_1) \vee \ldots \vee (p \,\&\, q_n) && \text{(nDist)}\\
\sim(p_1 \vee \ldots \vee p_n) \quad &\Leftrightarrow \quad \sim p_1 \,\&\, \ldots \,\&\, \sim p_n && \text{(nDeM)}\\
\sim(p_1 \,\&\, \ldots \,\&\, p_n) \quad &\Leftrightarrow \quad \sim p_1 \vee \ldots \vee \sim p_n && \text{(nDeM)}
\end{aligned}$$

These n-ary forms may be proved from the binary forms of the theorems by recursion: first note that the theorem holds for n=1 and n=2; then show that given any natural number k, if the theorem holds for n=k it must also hold for n=k+1; it then follows that the theorem holds for all natural n (Why?). The details of these proofs are left as an exercise for the interested reader. It should be noted in passing that the method of *recursive proof* (sometimes given the misnomer "mathematical induction") is one of the most powerful techniques of logical deduction.

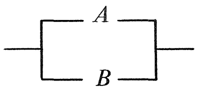
In setting out a natural deduction for network simplification, we shall begin by writing down the formula corresponding to the original network and write "ON" beside this. Besides being an abbreviation for "Original Network", the "ON" reminds us that from the PC point of view, the formula states the conditions necessary for the whole network to transmit or be "on". The deduction may be interpreted in terms of SC or PC: we suggest the latter, and that in either case "∼" be used in preference to " ′ ". Each deduction line after the first will be equivalent to the previous line according to the theorem noted in the justification column.

Example: Simplify the following network.



1. $(\sim A \,\&\, B) \vee A \vee (\sim B \,\&\, A)$	ON
2. $(\sim A \,\&\, B) \vee [A \vee (A \,\&\, \sim B)]$	Com &, Assoc ∨
3. $(\sim A \,\&\, B) \vee A$	Abs
4. $A \vee (\sim A \,\&\, B)$	Com ∨
5. $(A \vee \sim A) \,\&\, (A \vee B)$	Dist
6. $\top \,\&\, (A \vee B)$	LEM
7. $A \vee B$	Id

In PC terms, the simplified formula "$A \vee B$" says that the network transmits iff switch A transmits or switch B transmits. The simplified network is shown opposite.



Design of Switching Circuits

With practice it is often possible to "see" a working design for a particular task in terms of a circuit diagram; we need then only look for an optimum simplification of the circuit. If solving directly in terms of a circuit diagram proves difficult however, the following "brute force" method may be used:

1. List the alternatives for which the circuit will conduct and *disjoin* these;
2. Simplify this disjunction as much as possible;
3. Draw the circuit for the simplified formula.

Example: Three people A, B and C are to vote on an issue. Each has a switch which may be closed or opened (corresponding to a YES or NO vote). Design a switching circuit (the simpler the better) such that a light will come on if and only if the issue is passed by a majority vote.

For a majority vote we need at least two to vote yes. So the alternative ways of having the light come on are: A and B vote YES;
A and C vote YES;
B and C vote YES.

(We read these as "At least A and B vote YES" etc. Thus the case where A, B and C all vote YES is included in each of these alternatives.)

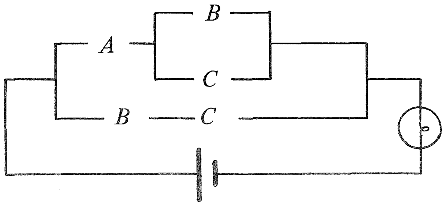
Using "A" as an abbreviation for "A closes his switch", and likewise for "B" and "C", we can say that the light will come on iff the following disjunction is satisfied:

$$(A \,\&\, B) \vee (A \,\&\, C) \vee (B \,\&\, C)$$

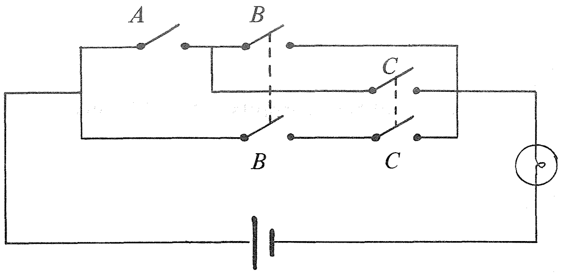
By Distribution this formula simplifies to:

$$[A \,\&\, (B \vee C)] \vee (B \,\&\, C)$$

Letting "A", "B" and "C" now denote switches, the required voting circuit may be drawn as shown:



In practice, switch A, one of the B switches and one of the C switches would be placed in front of the voters (one each); the other B and C switches would be linked into the circuit so as to match the states of the B and C switches flipped by the voters. A *primitive* way of arranging this is shown below, where the broken lines joining corresponding switches represent a physical (not electrical) connection.



Before going on to another example it will be useful to tie in the first step of our method, wherein we disjoin conducting alternatives, with the notion of disjunctive normal form. Recall that an elementary wff of PL is either a propositional letter or a negated propositional letter.

Definition: A formula is in *disjunctive normal form* (DNF) iff it is a disjunction of conjunctions of elementary wffs.

We allow elementary wffs to count as degenerate disjunctions and conjunctions, and permit n-place disjunctions and conjunctions. Thus each of the following is in DNF:

$$A, \quad A \vee B, \quad A \,\&\, \sim B, \quad \sim A \vee B \vee C, \quad \sim A \vee (A \,\&\, B \,\&\, C)$$

Here are some examples of formulae not in DNF:

$$\sim (A \,\&\, B), \quad \sim (A \vee B) \vee C, \quad \sim\sim A$$

Note that if a formula is in DNF, only propositional letters can be negated.

Note that in the previous circuit design example, the original formula "$(A \,\&\, B) \vee (A \,\&\, C) \vee (B \,\&\, C)$" is in DNF. For any design problem that we consider this will always be the case. In fact it is always possible in principle to convert any propositional formula to an equivalent one in DNF. This is intuitively obvious if we reflect that a truth table can be constructed for any propositional formula, that each row of the matrix is a conjunction of elementary wffs, and that the *formula is equivalent to a disjunction of the matrix rows for which the formula is true.* For instance, in the case of the formula $\sim (p \,\&\, q)$ since it is true on just rows 2 to 4 it must be equivalent to the disjunction of these rows i.e.

$$(p \,\&\, \sim q) \vee (\sim p \,\&\, q) \vee (\sim p \,\&\, \sim q)$$

which is of course in DNF. The switching circuit analogue of the truth table is the transmission table. In several types of design problems it is easiest to first draw up the transmission table for the circuit. A circuit formula in DNF can then be immediately read off from this table. This formula will then be simplified as much as possible before the final circuit diagram is drawn. This procedure will be illustrated in the next worked example.

p	q	$\sim (p \,\&\, q)$
1	1	0
1	0	1
0	1	1
0	0	1

Example: A hall light is to be controlled by two switches at opposite ends of the hall. Under all conditions, and no matter which switch is involved, a change in the state of the switch is to change the state (on/off) of the light. Design a switching circuit for this task.

Let us call the two switches A and B, and suppose that the light is on when both A and B are closed. This allows us to fill in the first row of the transmission table for the desired circuit as indicated. Since past history is irrelevant (the switches don't remember what has gone on before they are put in their current state) *whenever* A and B are closed the light will be on. Similarly, for any other given assignment of closure values to A and B there will be a unique transmission value for the light.

We can now fill in the rest of the transmission table by listing all the possible permutations of values for A and B in the matrix, and determining the values for the light by the following principle: a change in the value of A will change the light value; a change in the value of B will change the light's value. Beginning with row one, a change in B will make the light go off (row two); starting with row two, a change in A will make the light come on (row 4); starting with row 4, a change in B will make the light go off (row 3). We have now completed the transmission table.

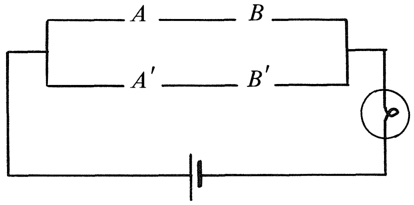
A	B	light
1	1	1
1	0	0
0	1	0
0	0	1

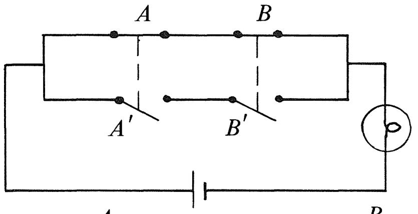
From this table we can now produce a formula in DNF for the circuit:

$$(A \,\&\, B) \vee (\sim A \,\&\, \sim B)$$

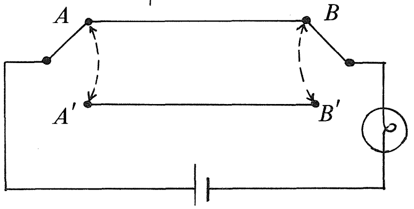
This formula does not simplify any further.

The circuit diagram in basic form is shown below on the left. The diagram on the right indicates a primitive way of arranging this, where the broken lines joining opposing switches denote non-electrical connections.





In practice, instead of four simple switches it would be more usual to employ two single-pole double-throw switches as shown opposite.



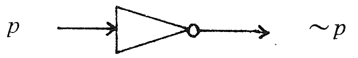
It is worth noting that transmission tables have several uses. For a start we know that two circuits will be equivalent iff they have identical transmission tables. Hence experimentally we could check that a simplified circuit was equivalent to the original by throwing the switches through all the possible permutations in the table matrix and checking that the output (in terms of on/off states of the circuit light) was the same in both cases. Again, suppose we are given a "black box" and asked to design a circuit to produce the same result: we could begin by testing the output for each row of the transmission table; once the table had then been filled in we would immediately have a circuit in DNF which would do the job; this could then be simplified where possible. Transmission tables can sometimes be useful for the simplification process, particularly when the original formula is a tricky one to reduce by the natural deduction technique; the DNF version of the network, which may be read straight from the table, may be easier to work with for simplification than the original formula.

Other applications of switching circuits include binary adders (we can treat 1 and 0 as binary numbers), and control circuits for industrial production lines. Complicated series-parallel circuits can often be simplified further in terms of networks which allow wye, delta and bridge circuits. If as well as switches we allow logic gates to feature as circuit elements, a virtually unlimited range of electronic applications of propositional logic opens up.

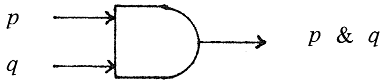
Logic Gates:

Integrated circuits used in computers contain large numbers of *logic gates* to assist in both decision making and computation. Here we look briefly at five such gates and note that PC may be applied to the simplification and design of gate networks in a similar way to switching networks. It is not necessary in this regard for you to understand their internal circuitry: we may treat the gates simply as "black boxes". "High" and "Low" are the only voltage states allowed. If we regard high and low voltages as corresponding to "true" and "false" the gates may be viewed as having the function of PC operators. The standard electronic symbols for the gates are shown below, with sample inputs on the left and the resultant output on the right. With the AND gate for instance the output voltage is high iff all its input voltages are high.

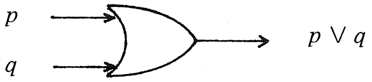
NOT Gate



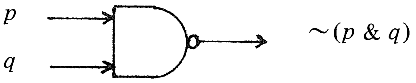
AND Gate



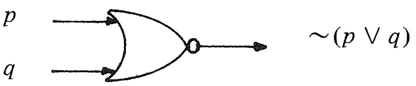
OR Gate



NAND Gate



NOR Gate



Of particular interest are the "Nand" and "Nor" gates. The *Nand* (*N*ot-*and*) operation is often symbolized in PC as "|" (Sheffer Stroke) and is defined as follows:

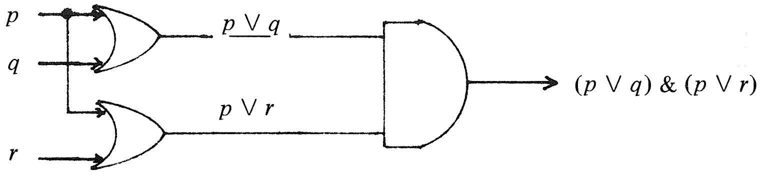
$$p \mid q =_{\mathrm{df}} \sim(p \,\&\, q)$$

The *Nor* (*N*ot-*or*) operation is usually symbolized in PC as "↓" (Peirce arrow) and is defined thus:

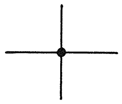
$$p \downarrow q =_{\mathrm{df}} \sim(p \vee q)$$

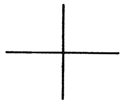
It may easily be shown that all the monadic and dyadic propositional operators (some of these you still have to meet in the next section) may be expressed in terms of "|". Hence it is possible, for any network of logic gates, to build an equivalent circuit out of Nand gates alone. Thus a microchip containing several Nand gates can be made to serve many different logical functions. Exactly the same thing may be said for the Nor gate.

Complicated networks may be built by connecting the output terminals of some gates to the input terminals of others e.g.,



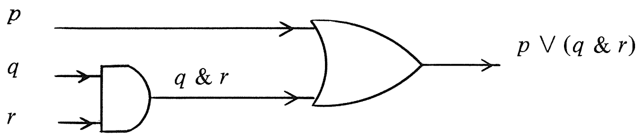
Here the final output is at a high voltage level iff (*p* ∨ *q*) & (*p* ∨ *r*) is true. Notice the convention we are adopting to indicate electrical connection:


connection


no connection

Hence in the network above, the input wire from p to the lower "Or" gate has no electrical connection with the input wire from q.

The final output above should have rung some logical bells in your mind. By Dist $\vee\&$ the formula is equivalent to $p \vee (q \,\&\, r)$. The circuit for this is:



Since for the same input this circuit will give the same output as the original, it may be used instead to do the same job. So the simplification procedure developed earlier for switching networks may also be applied to gate networks.

NOTES

In more formal treatments, Switching Calculus may be proved to be Boolean by taking S to be the set of transmission *states* $\{1, 0\}$. The operators are then state-operators and 1 and 0 are identities, in agreement with the tables for Exercise 9.3 Question 8.

We have taken a network to be simpler if it has fewer elements (switches or gates). Because of other technical reasons, a simpler circuit in this sense will not always be cheaper to construct.

The symbol names "Sheffer stroke" and "Peirce arrow" are in honour of two American logicians, Henry Maurice Sheffer (1883-1964) and Charles Sanders Peirce (pronounced "purse") (1839-1914).

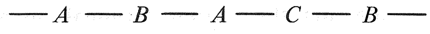
EXERCISE 9.5

1. For each of the following networks write the formula in PL, simplify this as much as possible and draw the simplified network. You may use any of the theorems in the chapter summary.

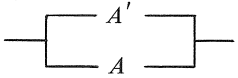
(a)

(b)

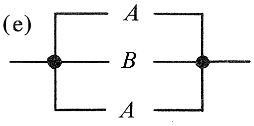
(c)



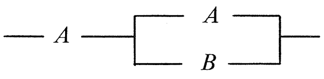
(d)



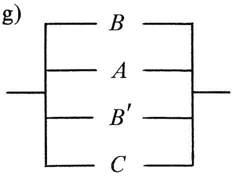
(e)



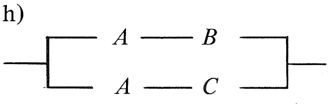
(f)



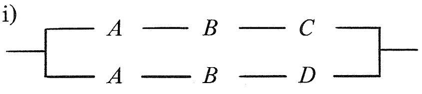
(g)



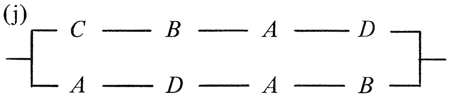
(h)



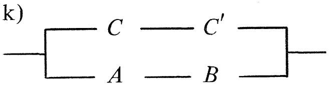
(i)



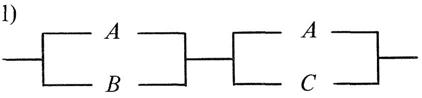
(j)



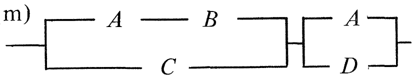
(k)



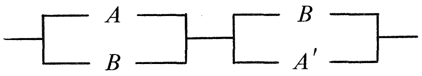
(l)



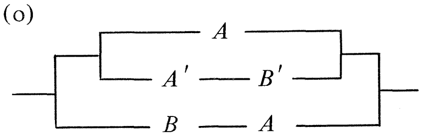
(m)



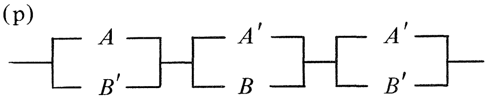
(n)



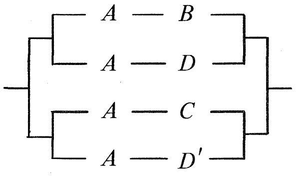
(o)



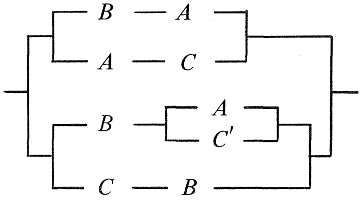
(p)



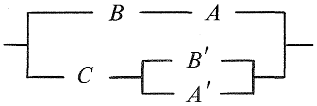
(q)



(r)

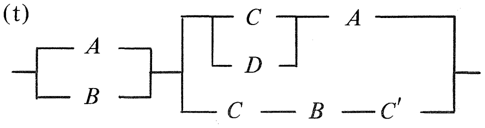


(s)

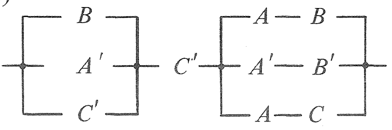


(*Hint*: Use DeM)

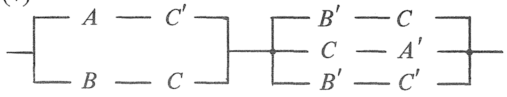
(t)



(u)



(v)



(*Hint*: By inspection of the diagram disjoin *possible* pathways from left to right)

2. Design and simplify a voting circuit for four people *A*, *B*, *C*, *D* such that the light will come on iff the motion is carried (i.e. iff a majority vote in favour).

3. As for Question 2, but *A* now has the power of veto (i.e. for the motion to be carried it is necessary that *A* vote in favour of it).

4. As for Question 3 but now there is a fifth person *E* on the voting panel.

5. A black box has two switches *A* and *B* and a light bulb on its surface. When the switches are thrown through all the permutations, the light behaves as shown in the transmission table below.

   Design a circuit which would produce the same output as the black box.

   Your circuit should be as simple as possible.

A	*B*	*light*
1	1	0
1	0	1
0	1	0
0	0	1

6. As for Question 5, but there are three switches A, B and C and the transmission table is as shown opposite.

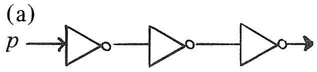
A	B	C	*light*
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	0

7. A large room has three doorways and one central light. Three switches A, B and C (one near each door) control this light in the following way: when A, B and C are all in the "up" position the light is on: thereafter any change in state of any of the three switches will change the state of the light. Each switch has just two possible states: "up" and "down". A fourth switch D is now added as a master control. Unless D is in the "up" position, the light cannot come on at all. Design a ciruit to do this job, and simplify it as much as possible.

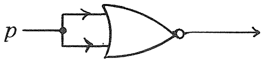
8. Use truth tables, trees or MAV to show that | and ↓ are commutative but *not* associative.

9. Simplify the following networks by determining the output formula, simplifying this as much as possible and drawing the network for the simpler formula.

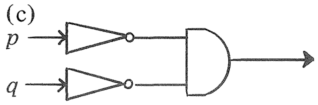
(a)



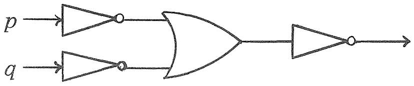
(b)



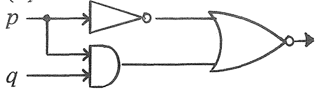
(c)



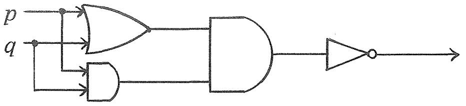
(d)



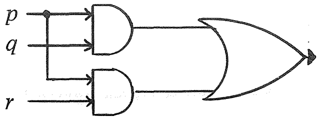
(e)



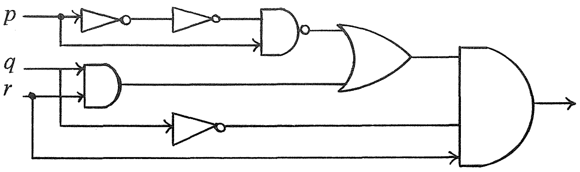
(f)



(g)



*(h)



*10. This one is for electronics enthusiasts. ICs containing several logic gates are available for less than $1.

(a) Construct a circuit with ICs, LEDs, torch cells and matrix board to illustrate the function of some logic gates. Let the "on" state of the LED correspond to "true" and the "off" state to "false".

(b) Use a group of nand gates or a group of nor gates to construct other gates.

(c) Construct appropriate circuits and demonstrate DeMorgan's Laws.

(d) Investigate how logic gates may be used to construct circuits for doing arithmetic on binary numbers. Look up the function of "half-adders" and "full-adders" and see if you can design these adders yourself using logic gates. Then try subtraction, multiplication and division.

9.6 FURTHER PROPOSITIONAL OPERATORS

In §2.3 truth-tabular semantics were given for one monadic propositional operator ($\sim$) and five dyadic propositional operators (&, $\vee$, $\supset$, $\equiv$, $\not\equiv$). Two further dyadic operators ($|$, $\downarrow$) were introduced in §9.5. But this does not exhaust the possibilities.

If we let $*$ denote a monadic propositional operator, its defining truth table is obtained by filling each of the 2 rows with one of 2 truth values. Clearly there are 4 (= 2 x 2) ways of doing this, so there are 4 monadic propositional operators. We symbolize these as shown.

p	$*p$
1	
0	

p	$\mathbf{V}p$	$\mathbf{I}p$	$\sim p$	$\mathbf{F}p$
1	1	1	0	0
0	1	0	1	0

The *verum* operator **V** always yields the value true, and the *falsum* operator **F** always yields false. The *identity* operator **I** yields a value identical with its operand.

If we let $*$ denote a dyadic propositional operator, its defining truth table has 4 rows each of which may be filled in with one of 2 truth values. So there are 16 (= 2^4) dyadic propositional operators. We symbolize these as shown: in the table we assume that p is to the left and q to the right of the operator in each case.

p	q	$p * q$
1	1	
1	0	
0	1	
0	0	

p	q	$\mathbf{V}$	$\vee$	$\subset$	$\curvearrowleft$	$\supset$	$\curvearrowright$	$\equiv$	&	$\vert$	$\not\equiv$	$\not\curvearrowright$	$\not\supset$	$\not\curvearrowleft$	$\not\subset$	$\downarrow$	$\mathbf{F}$
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
1	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
0	1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0

The dyadic verum and falsum operators are rarely used. The following readings may be used for the others:

$p \vee q$	p wedge q	p and/or q
$p \subset q$	p converse hook q	p if q
$p \curvearrowleft q$	p left arrow q	p, whether or not q
$p \supset q$	p hook q	if p then q
$p \curvearrowright q$	p right arrow q	q, whether or not p
$p \equiv q$	p tribar q	p iff q
p & q	p ampersand q	p and q
$p \mid q$	p stroke q	not both p and q
$p \not\equiv q$	p slash tribar q	p or q but not both
$p \not\curvearrowright q$	p slash r. arrow q	not q, whether or not p
$p \not\supset q$	p slash hook q	p but not q
$p \not\curvearrowleft q$	p slash l. arrow q	not p, whether or not q
$p \not\subset q$	p slash c. hook q	not p, and q
$p \downarrow q$	p Peirce arrow q	neither p nor q

It has already been noted that PC can be set out in terms of just one of $|$ or $\downarrow$: each of these operators is said to be *adequate* for PC. Certain pairs are also adequate: $\sim$, &; $\sim, \vee$; $\sim, \supset$; $\sim, \subset$; $\sim$, $\not\supset$; $\sim$, $\not\subset$.

Besides monadic and dyadic operators, there are triadic (3-adic), tetradic (4-adic), and in general n-adic propositional operators. One important triadic operator is Alonzo Church's *conditioned disjunction*: $\mathbf{C}[p,q,r]$ may be read as "p or r according as q or not q" or "If q then p, else r": this has the value of p when q is true, and the value of r when q is false.

Finally we consider one operator of no fixed adinity (it can operate on one or more propositions). It is the *just one of* operator, which we symbolize as $\mathbf{J}(\ ,\ ,\dots)$. It is useful when we want to make clear that just one of several propositions is true. Note that $\not\equiv$ by itself cannot do this except for the case of two propositions.

$\mathbf{J}(p)$	is equivalent to $\mathbf{I}p$
$\mathbf{J}(p, q)$	is equivalent to $p \not\equiv q$
$\mathbf{J}(p,q,r)$	is equivalent to $(p \not\equiv q \not\equiv r) \,\&\, \sim(p \,\&\, q \,\&\, r)$

NOTES

For details on conditioned disjunction see Church's *Introduction to Mathematical Logic Vol 1* §24. We have written "$\mathbf{C}[p,q,r]$" where Church writes "$[p,q,r]$ to clarify the position of the operator's column in truth table work. This operator bears a resemblance to the conditional execution of computer programming: If p then i_1 else i_2 (however, while p denotes a proposition, i_1 and i_2 typically denote instruction sequences: i_1 is executed iff p is true; i_2 is executed iff p is false).

Polish notation for PC will be discussed in §9.8. Some other notations in use are given below.

Our notation	Some other notations
$\sim p$	$p,\ -p,\ -p,\ p'$
$p \,\&\, q$	$p \bullet q,\ p \wedge q,\ pq$
$p \vee q$	$p + q$
$p \supset q$	$p \rightarrow q,\ p \Rightarrow q$
$p \equiv q$	$p \leftrightarrow q,\ p \Leftrightarrow q,\ p = q$
$p \not\equiv q$	$p \veebar q,\ p \vee q$
1	t, T, W
0	f, F, ⊥

EXERCISE 9.6

1. Test the following to show which are tautologies, which contradictions, and which are contingencies.

   (a) $\sim p \equiv (p \supset \mathbf{F}p)$
   (b) $(p \,\&\, q) \equiv ((p \not\subset q) \not\subset q)$
   (c) $\sim p \equiv (p \not\subset \mathbf{V}p)$
   (d) $(p \curvearrowright q) \not\equiv (q \curvearrowleft p)$
   (e) $(p \,\&\, \mathbf{F}q) \equiv (q \not\curvearrowright \mathbf{F}p)$
   (f) $\mathbf{J}(p \supset p,\ q \supset q)$
   (g) $\mathbf{J}(p \equiv q,\ \sim p)$
   (h) $\mathbf{J}(p \equiv q,\ \mathbf{J}(p, q))$
   (i) $((p \downarrow p) \downarrow q) \mid ((p \downarrow p) \downarrow q)$
   (j) $(p\ \mathbf{F}\ q) \supset (p\ \mathbf{V}\ q)$

2. Set out a defining truth-table for $\mathbf{C}[p, q, r]$. Place the main-column of values under the $\mathbf{C}$.

3. Test the following to show which are tautologies, which contradictions, and which are contingencies.

   (a) $\mathbf{C}[p, p, p]$
   (b) $\mathbf{C}[\mathbf{C}[p, q, r], \mathbf{C}[p, r, q], \mathbf{C}[q, r, p]]$
   (c) $\sim\mathbf{C}[\mathbf{F}q, \mathbf{J}(q, p, \sim q), p \not\subset q]$
   (d) $\mathbf{C}[p, q, \sim p] \vee \mathbf{C}[\sim p, q, p]$

*4. For each of the following write out two formulae which are tautologically equivalent to the formula listed, one in which the only operator is $|$, and the other in which the only operator is $\downarrow$.

   (a) $p \supset q$
   (b) $p \equiv q$
   (c) $p \not\equiv q$
   (d) $p \supset (q \supset p)$
   (e) $(\sim p \supset q) \supset (\sim q \supset p)$

*5. For each of the following formulae write down which of the 16 dyadic propositional operators will yield a tautology when substituted for *.

   (a) $p * p$
   (b) $p * q \;\; .{\equiv}. \;\; q * p$
   (c) $p * (q * r) \;\; .{\equiv}. \;\; (p * q) * r$
   (d) $p * (q * r) \;\; .{\supset}. \;\; (p * q) * (p * r)$
   (e) $(p * q) * (p * r) \;\; .{\supset}. \;\; p * (q * r)$

9.7 REDUCING PARENTHESES

There are several conventions in existence for reducing the number of parentheses in formulae of PC. In this section we look at two such conventions. You will recall (§2.3, §3.2) that our practice has been to give $\sim$ priority over the other operators, and to allow parentheses to be omitted when they are outermost or when they are redundant because of associativity. We have also used dots in place of brackets to highlight the main operator of a formula or sub-formula. This dot notation, which we call *Russell's dot notation* after Bertrand Russell, may be extended to completely eliminate the need for brackets. The basic idea is that the formula's main operator should have adjacent to it (on either side) a group of dots greater in number than that for any other operator in the formula. For example, in

$$p \supset .q \supset r \;\; :{\supset}: \;\; p \supset q. \supset .p \supset r$$

the main operator has a group of two dots on each side. The main operator of the antecedent has a dot on its right side, and the main operator of the consequent has one dot on each side. These dots may be replaced by parentheses as follows:

$$\begin{array}{l} p \supset .q \supset r \;\; :{\supset}: \;\; p \supset q. \supset .p \supset r \\ (p \supset .q \supset r) \supset (p \supset q. \supset .p \supset r) \\ (p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r)) \end{array}$$

The largest dot group adjacent to the main operator indicates by its number of dots the depth of nesting of the operators. In the above example the nesting is two levels deep; in the following example the nesting is four levels deep.

$$\begin{array}{lllllll}
\sim ::p & \supset .q \supset r & .: \supset :. & p \supset q & :\supset: & q \supset s. & \supset .q \supset q \\
\sim \;(p & \supset .q \supset r & .: \supset :. & p \supset q & :\supset: & q \supset s. & \supset .q \supset q) \\
\sim ((p & \supset .q \supset r\,) & \supset (\; & p \supset q & :\supset: & q \supset s. & \supset .q \supset q)\,) \\
\sim ((p & \supset (q \supset r)\,) & \supset (\, & (p \supset q) & \supset & (q \supset s. & \supset .q \supset q)\,)\,) \\
\sim ((p & \supset (q \supset r)\,) & \supset (\, & (p \supset q) & \supset & (\,(q \supset s) & \supset (q \supset q)\,)\,)\,)
\end{array}$$

Another way of reducing parentheses is to adopt a more detailed *priority convention* for evaluating the operators. Here is one such ranking:

$$\left\downarrow \begin{array}{l} \sim \\ \& \\ \vee \\ \supset,\ \equiv,\ \not\equiv \end{array}\right.$$

Operators of equal priority in the same nesting level are evaluated left to right. The following examples show how parentheses may be inserted for formulae written according to this order convention.

$$\begin{array}{rcl@{\qquad\qquad}r}
\sim p \supset q \vee r \,\&\, s & \equiv & r \,\&\, s & p \supset r \supset s \supset p \\
\sim p \supset q \vee (r \,\&\, s) & \equiv & (r \,\&\, s) & (p \supset r) \supset s \supset p \\
\sim p \supset (q \vee (r \,\&\, s)) & \equiv & (r \,\&\, s) & ((p \supset r) \supset s) \supset p \\
(\,\sim p \supset (q \vee (r \,\&\, s))) & \equiv & (r \,\&\, s) &
\end{array}$$

The above convention is adopted by Alonzo Church, who uses it in combination with his own dot notation (different from Russell's): he also uses *"pq"* in place of our *"p & q"*. While those authors who opt for a priority convention among dyadic operators typically place $\sim$, &, $\vee$ in the order shown above, they often differ in ordering the other operators. A few authors (e.g., Hilbert, Bernays, Kleene) give $\vee$ a higher priority than &, but this conflicts with the order now commonly adopted in computer programming languages (viz. NOT, then AND, then OR).

EXERCISE 9.7

1. Rewrite each of the following using parentheses instead of dots (Russell's dot notation is used).

   (a) $\sim p \vee q \;.\!\equiv\!.\; p \supset q$
   (b) $p \supset q \;.\!\supset p : \supset : p \;.\!:\equiv:.\; p \vee \sim p$
   (c) $\sim : p \supset . \;q \,\&\, r \;.\!: \equiv :.\; p \,\&\, q \;.\!\supset\!.\; r \vee q$
   (d) $p \supset q \;.\!\supset\!.\; q \supset p : \equiv : r \vee s$
   (e) $\sim : p \supset q \;.\!\vee\!.\; q \supset p$

2. Rewrite each of the following, with parentheses inserted in accordance with the priority convention for dyadic operators mentioned in this section.

   (a) $\sim p \,\&\, \sim q \supset \sim p \vee q$
   (b) $p \,\&\, q \equiv q \,\&\, p$
   (c) $p \vee q \not\equiv r \,\&\, q \supset s$
   (d) $\sim (p \vee q) \equiv \sim p \,\&\, \sim q$
   (e) $p \supset q \equiv r \equiv s$

*3. Investigate Church's dot notation as set out in his *Introduction to Mathematical Logic Vol 1* pp 74-80, and rewrite the following formulae (of his notation) into our usual notation with no dots.

(a) $\sim p \supset \ .q \supset p$
(b) $[q \supset \ .p \supset p] \supset \sim q$
(c) $p \supset [q \supset p] \supset .p \supset q \supset .p \supset p$
(d) $p \equiv q \ .\sim r$
(e) $r \equiv pq \supset .p \supset .s \supset [q \equiv p{\sim}r] \supset \sim s$

9.8 POLISH NOTATION

The *Polish Notation* for PC is a completely parenthesis-free notation invented by Jan Łukasiewicz (1878-1956). While lower case letters are used for propositional variables and constants, all the operators are symbolized by capital letters. So formulae in Polish notation may be typed on an ordinary type-writer.

Negation is symbolized as N, so that $\sim \alpha$ is rendered in Polish as Nα. Thus there is little difference as regards this monadic operator. The big difference is with the dyadic operators. In our notation (known as Peano-Russellian notation) the dyadic operators are written in *infix* position (i.e. in between the operands). In Polish notation the dyadic operators are written in *prefix* position (i.e. they precede their operands). For example instead of $(\alpha \mathbin{\&} \beta)$ we have K$\alpha\beta$, and instead of $(\alpha \supset \beta)$ we have C$\alpha\beta$. Note that the antecedent of the conditional is still to the left of the consequent. The letters used for the seven most important dyadic operators are shown below:

&	K	(Konjunction)
$\vee$	A	(Alternation)
$\supset$	C	(Conditional)
$\equiv$	E	(Equivalence)
$\not\equiv$	J	
$\mid$	D	
$\downarrow$	X	

Using "WPN" as an abbreviation for "Wff of Polish Notation" the following formation rules may be laid down.

Any single lower case letter is a WPN (B)

If α is a WPN so is Nα (RN)
If α and β are WPN so is $*\alpha\beta$ (R*)
where * is one of the letters V,A,B,I,C,H,E,K,D,J,G,L,F,M,X,O
If α is a WPN it is so because of the above rules (T)

The 16 dyadic operators V, ..., 0 correspond in order to the 16 dyadic operators defined in the table of §9.6. The formation rules ensure that there is no ambiguity in Polish notation e.g., NCpq is $\sim(p \supset q)$ while CNpq is $(\sim p \supset q)$. In any Polish wff other than a simple lower case letter, the left-most symbol is the main operator. Truth tables may be filled in by starting with the right-most symbol and working our way to the left e.g.,

p	q	E	A	N	p	q	C	p	q
1	1	1	1	0	1	1	1	1	1
1	0	1	0	0	1	0	0	0	1
0	1	1	1	1	0	1	1	1	0
0	0	1	1	1	0	0	1	0	0
		8	7	6	5	4	3	2	1
		↑							

Columns 1 and 2 are copied from the matrix; 3 follows from 1 and 2; 4 and 5 are copied; 6 follows from 5; 7 follows from 6 and 4; and 8 follows from 3 and 7. One simple way to translate a Polish notation formula into Peano-Russellian notation is to set up an assembly line for the WPN and then reproduce this with Peano-Russellian rules.

Polish notation can be simply reversed to get what is known as *Reverse Polish*. For example, N*Cpq* becomes *qp*CN in Reverse Polish. The interested reader is left to work out the formation rules for Reverse Polish.

EXERCISE 9.8

1. Convert each of the following to Polish Notation.
   - (a) $p \supset (q \supset p)$
   - (b) $(p \,\&\, q) \supset (p \vee q)$
   - (c) $(\sim p \vee q) \equiv (p \supset q)$
   - (d) $\sim(p \vee q) \equiv (\sim p \,\&\, \sim q)$
   - (e) $(p \supset (p \supset q)) \supset q$

2. Convert each of the following to Peano-Russellian notation.
   - (a) C K C *p q* C *q r* C *p r*
   - (b) C C A *p q r* K C *p r* C *q r*
   - (c) C C C *p q r* C C *r p p*
   - (d) C C *p q* C N *q* N *p*
   - (e) E K J J *p q r* N K K *p q r* K A A *p q r* K N K *p q* K N K *q r* N K *p r*

9.9 SUMMARY

To solve a *puzzle* that asks for an alternative compatible with a set of conditions, *truth tables* or *recording grids* may help. With the former method, essential propositions are symbolized and a table constructed with a full or restricted matrix, then rows that fail to satisfy the conditions are eliminated. With the latter method, items or properties to be matched are entered as coordinates of a rectangular grid: deduced matching or non-matching of a coordinate pair is indicated by entering a "1" or "0" in the cell of those coordinates; with one-to-one matching, the entry of "1" in a cell allows "0" to be entered in all other cells of the same row or column.

A set S is *closed* under a binary operation $*$ iff $x * y \in S$ for all $x, y \in S$. The element e is an *identity* for $*$ iff $x * e = x$ for all $x \in S$. A system $\langle S, +, \circ, ' \rangle$ is *Boolean* iff: S is closed under $+$, $\circ$ and $'$; $+$ and $\circ$ commute and distribute; S contains 0 and 1 which are identities for $+$ and $\circ$; and each element x has a complement x' such that $x + x' = 1$ and $x \circ x' = 0$. For each Boolean theorem there is a *dual* theorem in which $+, \circ, 0, 1$ have been uniformly substituted for $\circ, +, 1, 0$. *Cayley tables* may be used to test for closure (no new element), commutativity (symmetry about the main diagonal), and idempotence (main diagonal matches table heading).

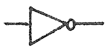
Though "ordinary" algebra is not Boolean, both *PC* and *Set Theory* are. The isomorphism between the systems may be used to translate results from one system to another according to the symbolic correspondence below (here "UBS" and "ST" abbreviate "Uninterpreted Boolean System" and "Set Theory"):

UBS	$x, y, \ldots$	$+$	$\bullet$	$'$	1	0	$=$
PC	$p, q, \ldots$	$\vee$	$\&$	$\sim$	T	F	$\Leftrightarrow$
ST	$A, B, \ldots$	$\cup$	$\cap$	$'$	$\mathcal{E}$	$\{\}$	$=$

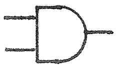
A formula in set theory notation may be tested for theoremhood within ST (e.g., by *Venn diagrams*), or by using PC methods on the corresponding PC formula.

Switching Calculus is also Boolean, with transmission values corresponding to truth values, switch opposition to $\sim$, series connection to &, parallel connection to $\vee$, an unbroken wire to the identity T, and a broken wire to the identity F. Switching networks may be *simplified* by replacing the network formula with a simpler, logically equivalent one, obtained by PC methods (e.g., natural deduction or tables). Circuits for particular tasks may often be *designed* by writing the network formula in *disjunctive normal form* (a disjunction of conjunctions of elementary wffs) then simplifying this.

Electronic devices known as *logic gates* may be constructed which perform the logical operations indicated below. Gate networks may be simplified by PC methods applied to the network formula. Each of the *nand* and *nor* operators, $\mid$ and $\downarrow$, is adequate for defining the other propositional operators. $p \mid q$ means $\sim(p \,\&\, q)$ and $p \downarrow q$ means $\sim(p \vee q)$.



NOT



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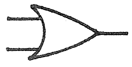

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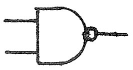
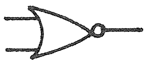


NAND



NOR

As regards *propositional operators*, there are 4 monadic and 16 dyadic. Operators of higher adinity (e.g., "conditioned disjunction" **C**[, ,]) or variable adinity (e.g., "just one of " **J**(, ...)) may be defined.

The number of *parentheses* in a PC formula may be reduced by use of *dots*, or by adopting a *priority convention* among dyadic operators and evaluating *left to right*. Various different conventions exist in this regard. A third way of reducing parentheses is to adopt the *Polish notation* wherein the operators are represented as capital letters placed in prefix position e.g., Np for $\sim p$, Kpq for $p \,\&\, q$, Apq for $p \vee q$, Cpq for $p \supset q$ and Epq for $p \equiv q$.

The following PC theorems are of particular use in applying natural deduction to set theory, switching networks and logic gate networks. Those asterisked have a useful *n*-ary form.

$p \vee q$	$\Leftrightarrow$	$q \vee p$	Com	$\sim(p \vee q)$	$\Leftrightarrow$	$\sim p \,\&\, \sim q$	DeM*
$p \,\&\, q$	$\Leftrightarrow$	$q \,\&\, p$	Com	$\sim(p \,\&\, q)$	$\Leftrightarrow$	$\sim p \vee \sim q$	DeM*
$p \vee \mathsf{F}$	$\Leftrightarrow$	p	Id	$p \vee p$	$\Leftrightarrow$	p	Idem*
$p \,\&\, \mathsf{T}$	$\Leftrightarrow$	p	Id	$p \,\&\, p$	$\Leftrightarrow$	p	Idem*
$p \vee (q \,\&\, r)$	$\Leftrightarrow$	$(p \vee q) \,\&\, (p \vee r)$	Dist*	$p \vee (p \,\&\, q)$	$\Leftrightarrow$	p	Abs
$p \,\&\, (q \vee r)$	$\Leftrightarrow$	$(p \,\&\, q) \vee (p \,\&\, r)$	Dist*	$p \,\&\, (p \vee q)$	$\Leftrightarrow$	p	Abs
$p \vee \sim p$	$\Leftrightarrow$	T	LEM	$\sim \mathsf{T}$	$\Leftrightarrow$	F	NT
$p \,\&\, \sim p$	$\Leftrightarrow$	F	LNC	$\sim \mathsf{F}$	$\Leftrightarrow$	T	NF
$\sim\sim p$	$\Leftrightarrow$	p	DN	$p \vee \mathsf{T}$	$\Leftrightarrow$	T	DT
$p \vee (q \vee r)$	$\Leftrightarrow$	$(p \vee q) \vee r$	Assoc*	$p \,\&\, \mathsf{F}$	$\Leftrightarrow$	F	CF
$p \,\&\, (q \,\&\, r)$	$\Leftrightarrow$	$(p \,\&\, q) \,\&\, r$	Assoc*				

(To celebrate the end of Part One, the following PC crossword is presented as the Chapter Puzzle.)

PC CROSSWORD

1	2	3	4		5		6		■	7	8
■	9			■		■		■	10		
■	11			12	■	13		14	■	■	
■	15		■	16	17	■	18			19	
20			21			■	22		■		■
■		■		■		■		■	■	23	
24		■	25	26	■	■		■	27	■	■
■	28	29	■	30	31	■	32			■	33
34			35					■	36		
■		■	37		■	■	38			■	

Clues:

Across:

1. Study of the meanings of symbols
7. Not not
9. Half of 7 across
10. $p \supset \sim p . \supset . \sim p$
11. Climb this to find truth
13. Operator for conjunction
15. Santa repeats this
16. Regarding
18. Valid with true premises
20. Study of symbols as uninterpreted objects
22. Latin and French for 13
23. Same as 24
24. Same as 16
25. Freudian concept
28. $(p \supset q) \& p . \supset . q$
30. This is it!
32. I
34. This type of procedure gives a finite mechanical solution
36. A logical bark
37. Part 2 of this book
38. Truth tables or truth trees?

Down:

2. These have tacit premises
3. This person doesn't study logic
4. Consumed
5. Singular of 38
6. On the right of a hook
7. Father
8. Can handle all our operators by itself
12. Era
14. Like a bracket
17. $(p \& q) \supset r . \supset . p \supset (q \supset r)$
19. Entitled to make the same boast as 8
21. Prefix in syntactical name of material equivalence symbol
26. $p \& (q \vee r) . \equiv . (p \& q) \vee (p \& r)$
27. These are horizontal
29. Part 1 of this book
31. A different type of tree
33. Material equivalence
35. Studying logic improves this

Part Two

Quantification Theory

10 A Richer Language

10.1 INTRODUCTION

We have already seen in §4.4, that many arguments and propositions are complex in ways that cannot be accounted for in propositional logic alone.

Consider the argument:

All space movies are fascinating
Star Wars is a space movie
∴ *Star Wars* is fascinating

This is obviously valid, since given that the premises are true the conclusion must be true. However, if we translated it into PC we would get something like

$$\begin{array}{ll} & F \\ & S \\ \hline \therefore & W \end{array}$$

F = All space movies are fascinating
S = *Star Wars* is a space movie
W = *Star Wars* is fascinating

This has, as its explicit PL-form, the form

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & r \end{array}$$

This form is invalid in PC. It is only when we use a possible-truth table that we discover the argument to be valid. PC alone has not revealed a valid form.

The validity of the argument flows from an argument form in which account is taken of the terms "all", "space movie" and "fantastic". Such a form would be something like

All *S* are *F*
a is *S*
∴ *a* is *F*

But PC has no way of taking into account the sub-propositional elements which appear as *S, F, a* and *all*.

There are also logical relationships which cannot be explained in terms of propositional logic alone. It is quite clear that if (1) is true, then so is (2).

John is a farmer (1)
There is at least one farmer (2)

(1) necessarily implies (2), but not conversely.

It is also clear that (3) and (4) are contradictory.

All roses are red. (3)
Some rose is not red. (4)

These relationships of necessary implication and contradictoriness cannot be detected in PC alone. To handle such cases we introduced the additional intuitive method of possible-truth tables.

We will now begin to set out a logic which can deal with the argument above, and the relationships between (1) and (2), and (3) and (4), wholly within the system. The logical system is known as *Monadic Quantification Theory* (MQT).

MQT uses a language which is richer than PL. The language, not unexpectedly, is called *Monadic Quantificational Language* (MQL). MQL is part of a broader language called *Quantificational Language* (QL). MQT is part of a wider logical system called *Quantification Theory* (QT). When we have finished looking at MQT we will go on to QT.

MQT (and QT) is really an extension of the Propositional Calculus studied in Part One, i.e. MQT *subsumes* PC. In this chapter we look first at some important features of English, and then introduce the new logical language. In subsequent chapters we consider translation, validity, necessary truth, and the use of truth-trees and natural deduction.

NOTES

The first quantificational logic was constructed by Aristotle in *Prior Analytics*. But the system was extremely restricted. Quantification Theory, in the general sense, was first set out by Gottlob Frege (1848-1925) in his *Begriffsschrift*. Other names for QT are: (first-order) predicate calculus; (first-order) functional calculus; lower predicate calculus; predicate logic.

The symbolism used here, and in general use today, is not Frege's symbolism, but Russell's modification of the notation used by Peano. See *Introduction to Mathematical Logic* Vol I by Alonzo Church and *The Development of Logic* by W & M Kneale.

10.2 SINGULAR REFERENCE, PROPERTIES AND QUANTITY

Before we set out the formal language of MQL, we will look in some detail at propositions expressed in English. We will delve into the internal structure of propositions such as

Alan is a farmer
and All cats are mammals.

At first we focus our attention on *singular reference* i.e. reference to a quite specific entity. For example, in:

Alan is a farmer.

there is reference to one quite specific individual: Alan. In some propositions there is more than one such singular reference. For example, in the conjunction:

Bill and Carl are farmers.

there is singular reference to both Bill and Carl. In the cases above, singular reference has been expressed by *proper names*. But there are other ways of expressing singular reference. For:

The Lord mayor is a butcher.

singular reference has been expressed by the phrase "the lord mayor". Such phrases, beginning with the word "the" are called *definite descriptions.* Definite descriptions express singular reference because they purport to refer to a unique individual, entity or item.

Yet another way of expressing singular reference is to use *singular pronouns*. Consider the following two propositions:

I am doing logic
It is a square.

Some singular pronouns are:

I me you he she him her it that this

So, *proper names, definite descriptions* and *singular pronouns* are all singular terms. Such terms can be used to make singular reference, not only to persons, but also to places, times, animals, things, and all sorts of individual items, even items which do not exist. We will use the term *item* to cover all such specific persons, places, times, animals, things, etc.

In propositions, when there is singular reference, something is asserted about the *item* to which reference is made. In:

Alan is a farmer

there is reference to Alan, and the property of *being a farmer* is *predicated* of Alan.

When there is singular reference in propositions there will be *logical subjects* and a *logical predicates*. Logical subjects are expressed by the singular terms and the logical predicates are the properties or relations predicated of the items. In the proposition:

Alan is a farmer.

the logical subject is *Alan* and the logical predicate is *being a farmer*. In the proposition:

Sue loves Alan

the logical subjects are *Sue* and *Alan*, and the logical predicate is the relation *loves*. Note here that the grammatical object "Alan" still counts as a logical subject.

So far we have been considering singular reference, but consider the proposition:

Every one is happy.

There is no singular reference here. A *quantity* of people is indicated – *all.* No specific person is nominated. Such a proposition is *quantified.* There are many quantity phrases in English, e.g.: *most, many, a few, a couple, some, at least one.*

We are going to focus our attention on just two quantities:

Every
At least one

The phrases are known as *quantifiers*. The first is a *universal* quantifier, and the second is a particular, or *existential* quantifier.

Consider the following atomic propositions:

Everyone is happy
Everybody loves somebody sometime.

The first is *singly*-quantified, the second is *multiply*-quantified.

We look first at *singly-quantified* propositions containing a *universal quantification.* Consider (1):

Every thing is green. (1)

(1) is a *universally-quantified simple proposition.* (1) says that each and every thing, taken one at a time, has a certain property, the property of being green. In what follows we will use the phrase "Every thing" to mean "Each and every thing, taken one at a time."

Consider (2):

Every thing is non-green. (2)

(2) is a *universally-quantified negation.* Each and every thing lacks the property of being green. Consider (3):

Every thing is both green and spherical. (3)

(3) is a *universally-quantified conjunction.* It asserts that each and everything has both the property of being green and of being spherical. What a strange world it would be were (3) true. Consider (4):

Every thing is either green or spherical. (4)

(4) is a *universally-quantified disjunction.* But there is a possibility that (4) could be read in either of two ways:

Each and every thing, taken one at a time, is either
green or spherical (4a)
Every thing is green or every thing is spherical (4b)

(4a) is the universally-quantified disjunction. (4b) is a disjunction of universally-quantified simple propositions.

Consider (5)

Each and every thing, if spherical, is green (5)

(5) is a *universally-quantified conditional.* (5a) makes clear what a universally-quantified conditional asserts:

Each and every thing, taken one at a time, if spherical, is green. (5a)

It is important to note that the universal quantification in propositions (1) to (5) indicates something about each individual item.

Of the universally quantified propositions it is of some interest, and importance, to consider universally-quantified negations and conditionals in more detail. Look carefully at (6) and (7).

Every thing is non-green (6)
Nothing is green (7)

Clearly they both express the same proposition. "Nothing" is a term for universal negative quantification.

Care must be taken with (8):

Every thing is not green (8)

(8) can mean either that not everything is green or that nothing is green. It depends on emphasis.

The word "all" is very often used for universal quantification. (9) is the same proposition as (6) and (7).

All things are non-green (9)

But (10) suffers from the same ambiguity as (8)

All things are not green (10)

It is best to avoid the sentences used in (8) and (10) in favour of either the sentences used in (6), (7) and (9) or (11) and (12).

Not everything is green. (11)
Not all things are green. (12)

There is a wide range of sentences used to express universally-quantified conditionals. All of the following, (13) to (21), express the same as (5).

All green things are spherical. (13)
If anything is green then it's spherical. (14)
If something is green then it's spherical. (15)
Every green thing is spherical. (16)
Any green thing is spherical. (17)
Each and every thing, whatever else it is, if it's green
then it's spherical. (18)
Green things are spherical. (19)
Only spheres are green. (20)
Green things are all spheres. (21)

These are all universally-quantified conditionals. They assert that all those things with the property nominated antecedently (*being green*) also have the property set out in the consequent, (*being spherical*).

It is important not to confuse universally-quantified conditionals with universally-quantified conjunctions. In a universally-quantified conjunction it is asserted that every thing, literally everything, has both properties nominated in the conjuncts.

Universally-quantified conditionals are usually divided into two sub-groups. There are those like (13) to (21). They can be expressed in English by means of a sentence of the forms (22) or (23)

Every A is B (22)
All As are Bs (23)

The second group have negation as the main operator in the consequent. Examples are (24) to (26)

Every green thing is non-spherical (24)
Green things are not spherical (25)
All green things are non-spherical (26)

The same proposition is expressed by (27), (28) and (29):

No green thing is spherical (27)
No green things are spherical (28)
Nothing green is spherical (29)

These universally-quantified conditionals *with a negative consequent* can be expressed in English by means of sentences of forms (30) or (31)

Every *A* is *non-B* (30)
No *A* is *B* (31)

Universally-quantified conditionals without a negative consequent, of forms (22) or (23), are called *A propositions.* Universally quantified conditionals with a negative consequent, of forms (30) or (31), are called *E propositions. A* propositions are also known as *universal affirmative* propositions, and *E* propositions are known as *universal negative* propositions.

In ordinary language (23) and (31) are *contrary*, not contradictory. Use "rose" for *A* and "red" for *B* to see why.

We now turn to particular, or *existentially-quantified* propositions. The quantificational phrase which most accurately expresses this quantification is "at least one". Consider the following:

At least one thing is spherical (32)

(32) is an *existentially-quantified simple proposition.*

At least one thing is non-spherical (33)

(33) is an *existentially-quantified negation.*

At least one thing is both spherical and green. (34)

(34) is an *existentially-quantified conjunction.*

At least one thing is either spherical or green. (35)

(35) is an *existentially-quantified disjunction.*

At least one thing is such that if it's spherical then it's green (36)

(36) is an *existentially-quantified conditional.*

The word "some" is often used to express the same quantification as "at least one". So we can have:

Something is spherical (37)
Something is non-spherical (38)
Something is both spherical and green (39)
Something is either spherical or green (40)
Something is such that if it is spherical then it is green (41)

These express the same propositions as (32) to (36) respectively. When "some" is used in the singular, as in (37) to (41) then it means the same as "at least one". But "some" is often used in the plural as in (42) to (46).

Some things are spherical (42)
Some things are non spherical (43)
Some things are both spherical and green (44)
Some things are either spherical or green (45)
Some things if spherical are green (46)

These, (42) – (46), *necessarily imply* (37) to (41) respectively. In general we note that "some" in the plural sense necessarily implies "some" in the singular. Such plural uses of "some" often indicate more than one.

Our chief interest is in existentially-quantified negations, conjunctions, and simple propositions.

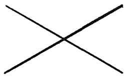
Consider first an existentially-quantified simple proposition (47), and a universally-quantified negation (48):

At least one thing is spherical (47)
Every thing is non-spherical (48)

These constitute a pair of contradictories. Consider also an existentially-quantified negation (49), and a universally-quantified simple proposition (50):

At least one thing is non-spherical (49)
Every thing is spherical (50)

Again we have a pair of contradictories. These contradictories can be set out in what is called a *square of opposition.*



Every thing is spherical	Everything is non-spherical
At least one thing is spherical	At least one thing is non-spherical

The universally-quantified propositions are across the top, the existentially-quantified across the bottom. The diagonal pairs are contradictory. From this we can also see some necessary equivalences. For example (51) is equivalent to (52)

Not every thing is spherical (51)
At least one thing is non spherical (52)

We now turn to the existentially-quantified conjunctions.

At least one thing is both spherical and green (53)

(53) can also be expressed by (59) to (56):

At least one sphere is green (54)
Some sphere is green (55)
One or more spheres are green (56)

We also note that (57) – (59) necessarily imply (53).

Some spheres are green (57)
Some spherical things are green (58)
Spheres are sometimes green (59)

In some existentially-quantified conjunctions, one of the conjuncts is negative, as in (60) and (61)

At least one sphere is not green (60)
Some spheres are not green (61)

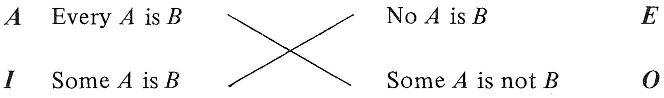
Existentially-quantified conjunctions are often separated into two groups. There are those which can be expressed by an English sentence of either form (62) or form (63):

At least one *A* is *B* (62)
Some *A* is *B* (63)

Such are known as ***I propositions.*** Others are expressed by English sentences of either form (64) or form (65):

At least one *A* is not *B* (64)
Some *A* is not *B* (65)

These are known as ***O propositions. A, E, I*** and ***O*** propositions can be arranged in a square of opposition where the diagonally opposite pairs are contradictory. For example:



Once again, as with the earlier square of opposition, we can set out necessary equivalences:

Not every A is B $\leftrightarrow$ Some A is not B

No A is B $\leftrightarrow$ It's not the case that some A is B

NOTES

We have adopted the term "item" in this text in order to make clear our adherence to an ontological neutrality. See *Exploring Meinong's Jungle* by R. Routley, especially pp. 5, 174-180. We do not adhere to the notion that singular terms have existential import.

Our preference is also to call the "at least one" quantifier a *particular quantifier*, but we have used the standard terminology. In later sections we will *not* assume that "at least one S is P" means the same as "These exists an S which is P". To that extent we are somewhat more optimistic about reforming the use of "$(\exists x)$" than is Routley.

The use of "A" and "I" with affirmative propositions derives from the Latin "*a*ff*i*rmo" = "I affirm". The use of "E" and "O" with negative propositions derives from the Latin "n*e*g*o*" = "I deny".

EXERCISE 10.2

1. Write out the singular terms in the following, and state what kind of singular term each is.
   (a) James is kind.
   (b) Susan is scholarly
   (c) If Tom is at school then Jill is at home.
   (d) The principal is on his way.
   (e) The Premier made a long speech.
   (f) Kirsty received the prize.
   (g) She was kind to him.
   (h) You promised me to give him the money.
   (i) The Wizard of Id has a problem.
   (j) The economy is in a bad way.

2. Write out the properties predicated of Jack and Jill in the following.
   (a) Jack and Jill fell down.
   (b) Jack was unsteady.
   (c) Jill was solicitous.
   (d) Jack was reassured.
   (e) Jill was beautiful.
   (f) Jack had a splitting headache.
   (g) Jill was deft with bandages.
   (h) Jack went to bed.
   (i) Jack and Jill were happy.
   (j) Jack slept.

3. Each of the following propositions pairs-off with one other with which it is necessarily equivalent. What are the five pairs?
   - (a) Every rose is a non-black object.
   - (b) At least one rose is a black object.
   - (c) Every thing is non-black.
   - (d) Not even one thing is non-black.
   - (e) Not even one thing is black.
   - (f) No rose is a black object.
   - (g) At least one rose is not a black object.
   - (h) At least one black object is a rose.
   - (i) Every thing is black.
   - (j) Not all roses are black objects.

4. Each of the following propositions pairs-off with one other of which it is contradictory. What are the five pairs?
   - (a) At least one radical is a student.
   - (b) Every non-radical is a non-student.
   - (c) All students are students.
   - (d) Not every non-radical is a non-student.
   - (e) Every student is a radical
   - (f) No student is a radical.
   - (g) Some radical is not a radical.
   - (h) At least one student is not a radical.
   - (i) Not even one radical is a student.
   - (j) Some student is a radical.

10.3 SYNTAX FOR MQL

In §2.2, PL was introduced from the point of view of syntax. There we set out rules for well formed formulae. In this section MQL is given similar treatment. If you find the going rather strange, don't give up. It is important to get at least a rough idea of what a formula of MQL is. In the next two sections you will find out what the symbols mean, and how to use them.

From now on, unless otherwise stated, "wff" will be taken to mean "wff of MQL". Before proceeding with the syntax of MQL, it would help if you revised the basic ideas covered in §2.2.

Primitive Symbols:

$p, q, r, s, t, \ldots$ $\sim, \&, \vee, \supset, \equiv, \not\equiv$ $(,)$	from PL
$a, b, c, d, e, \ldots$	Any number of small letters from the start of the alphabet.
$F, G, H, I, J, \ldots$	Any number of capital letters.
$x, y, z, w, v, \ldots$	Any number of small letters from the end of the alphabet.

$\forall$, $\exists$	Inverted A, reversed E, often read as "A" and "E".

In the discussion below the following terminology will be employed:

S	denotes any primitive drawn from F, G, ...
s	denotes any primitive drawn from either a, b, ... or x, y, ...
ν	denotes any primitive drawn from x, y, ...

A wff of MQL is a formula which obeys the following formation rules.

Formation Rules:

Basis Clauses:	$p, q, r, s, t, \ldots$ taken individually, are wffs	(B1)
	Anything of the form **S**s is a wff	(B2M)
Recursive Clauses:	If α is a wff, so is $\sim\alpha$	(R $\sim$)
	If α is a wff, so is $(\forall\nu)\alpha$	(R$\forall$)
	If α is a wff, so is $(\exists\nu)\alpha$	(R$\exists$)
	If α and β are wffs, so is $(\alpha \,\&\, \beta)$	(R&)
	If α and β are wffs, so is $(\alpha \vee \beta)$	(R$\vee$)
	If α and β are wffs, so is $(\alpha \supset \beta)$	(R$\supset$)
	If α and β are wffs, so is $(\alpha \equiv \beta)$	(R $\equiv$)
	If α and β are wffs, so is $(\alpha \not\equiv \beta)$	(R $\not\equiv$)
Terminal Clause:	If α is a wff, it is so because of the above clauses.	(T)

The only new rules are B2M, R$\forall$, and R$\exists$. Examples of the correct use of B2M are:

Fx, Fa, Gy, Gb, Hz, Ha

Wffs defined by B1 and B2M alone are *atomic* wffs. Examples are:

p, Fa, Gx, q

Wffs defined by B2M along, without using small letters from the end of the alphabet, are called *Subject Predicate atomic (SP atomic)* wffs. For example:

Fa, Fb, Ge, Ha, Hd.

Note carefully with R$\forall$ and R$\exists$ that *only* primitives from the list x, y, ... may be substituted for ν. So, whereas wffs of the type:

$(\forall x)\alpha$, $(\exists x)\alpha$, $(\forall y)\alpha$, $(\exists z)\alpha$, ...

are well-formed, expressions like:

$(\forall a)\alpha$, $(\exists b)\alpha$ are *not* wffs.

These rules may be used to construct wffs (as in the assembly line below) or to test whether an expression is a wff.

Example:	1. p	B1
	2. Fx	B2M
	3. $(\exists x)\, Fx$	2, R $\exists$
	4. $(p \supset (\exists x)\, Fx)$	1, 3, R $\supset$
	5. Gy	B2M
	6. $(Gy \equiv (p \supset (\exists x)\, Fx))$	4, 5, R $\equiv$
	7. $(\forall y)(Gy \equiv (p \supset (\exists x)\, Fx))$	6, R $\forall$

$\forall$ and $\exists$ are known as *quantifiers*. They always occur in *quantifications*:

$$(\forall x)\quad(\exists x)\quad(\forall y)\quad(\exists y)$$

Quantifications are somewhat like tildes. They are placed to the left of a wff to get a wff. The formula to which a quantification is added is known as the *scope* of that quantification:

The scope of $(\exists x)$ in $(\exists x)\,Fx$ is Fx
The scope of $(\forall y)$ in $(\forall y)(Fy \supset Gy)$ is $(Fy \supset Gy)$
The scope of $(\exists x)$ in $(p \supset (\exists x)\,Gx)$ is Gx
The scope of $(\forall x)$ in $((\forall x)\,Gx \supset (\exists x)\,Fx)$ is Gx

In every quantification there is one of the letters: $x, y, z, w, \ldots$. A quantification is a quantification with respect to (or wrt) that letter:

$(\exists z)$ is a quantification wrt z
$(\forall x)$ is a quantification wrt x

EXERCISE 10.3

1. Generate the following wffs from the formation rules, showing the justification for each step.
   (a) $(F\,x \supset G\,a)$
   (b) $(\sim F\,a \equiv G\,a)$
   (c) $\sim(F\,a \equiv G\,b)$
   (d) $((\exists x)\,F\,a \vee (\forall y)\,G\,y)$
   (e) $((\exists x)(\exists y)\,F\,x \equiv (\forall x)\,(\,G\,a \supset H\,b))$
   (f) $(\,\sim(\exists x)(\exists y)(\forall z)\,G\,x \supset (\forall x)(\exists z) \sim(F\,x\;\&\;H\,z))$

2. (a) Write down the scope of $(\forall x)$ in each of the following
   (i) $(\forall x)\,Fx \equiv Gx$
   (ii) $(\forall x)(Fx \vee (\exists y)\,Gy)$
   (iii) $(\forall x)(Fx \supset Gx)$
   (iv) $p \supset (\forall x)\,Fx$
   (v) $(\forall x)(p\;\&\;Gx)$

   (b) Write down the scope of $(\exists y)$ in each of the following:
   (i) $(\exists y)\,Fy$
   (ii) $(\exists y)\,p$
   (iii) $\sim((\exists y)\,Fy \supset Gx)$
   (iv) $(\exists y) \sim(Fy \vee Gy)$
   (v) $(\forall x)(\exists y)(Fy \supset Gx)$

3. Which of the following are SP atomic wffs?
   (a) Fa
   (b) p
   (c) Fx
   (d) Gd
   (e) $(\forall x)\,Fa$

10.4 SEMANTICS FOR INDIVIDUAL CONSTANTS AND PREDICATES

We must now set out a system for assigning truth-values to the wffs of MQL. We will do this by first giving the symbols of MQL a standard interpretation, and then showing how truth-values are calculated. The system is called *Monadic Quantification Theory* or MQT.

In order to set out MQT we first extend the notion of a *possible world*. We stipulate that each possible world contains at least one item. So, some possible worlds will have one item, some two items, some three, etc. Some will have infinitely many items and some finitely many items. For MQT we only ever *need* finitely many items, but we don't rule out infinitely many.

We will have one item worlds, two item worlds, three item worlds, and so on. The set of items in a world is usually called the *domain* of that world. Clearly, domains are non-empty sets of items.

The items in a world might be persons, creatures, plants, material objects, numbers, or any thing whatsoever. Such items can have or lack properties. It is this feature of items in possible worlds which we will use in MQT to determine whether or not *SP atomic* wffs are true or false in that possible world. We now turn to the symbols of MQL, give them an interpretation and show how this determines truth-value.

Looking back to our set of primitive symbols we note that those contained on the first three lines are familiar: apart from one extra use for parentheses these have exactly the same function in MQT as they had in PC.

Next we consider the symbols on the fourth line. These are the small letters at the beginning of the alphabet:

$a, b, c, d, e, f, g, \ldots$

These are called *individual constants.* Each may stand for a single item in a world. Individual constants are the logical version of proper names and other such singular terms. Note that each item in a world may have several individual constants referring to it, but an individual constant may refer to only one item.

In practice we may use any letter, except p, q, x, y, z, as an individual constant. In a dictionary we specify the names, or other singular terms, which are to be symbolized by various individual constants. Usually we pick the first letter of the name unless that letter has already been used, e.g.

a = Alan
b = Betty
c = the Chancellor
d = you
e = Europe
f = Friday
g = Anne

The capital letters on the fifth line of primitive symbols:

$F, G, H, \ldots$

are known as *predicate letters.* These are used to stand for properties which items have or do not have. The wff

$$Fa \tag{1}$$

means that the item a has the property F. So, if F means *being a farmer*, and a is as in the dictionary, (1) means

Alan is a farmer.

When we set out a dictionary for predicate letters, instead of using

F = being a farmer

we use

$$Fx = x \text{ is a farmer} \tag{2}$$

NOTE CAREFULLY, the "x" in this dictionary is just a blank space marker. It is *not* a symbol of MQL. We could have used (3) instead of (2)

$$F \ldots = \ldots \text{ is a farmer} \tag{3}$$

The "x" in the dictionary indicates that some symbol can be put in that spot, such as a or b or c etc, or, as it turns out below, even x, y, or z, etc.

We now set out a dictionary for predicate letters. As with individual constants, we may use any capital letter, and we usually pick the letter which is the first letter of the property word unless that letter has already been used. e.g.

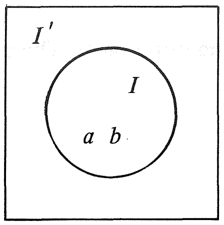
Fx = x is a farmer
Gx = x is a grocer
Hx = x is happy
Ix = x is an insect
Jx = x jumps
Kx = x is kind
Mx = x is a miser
Nx = x is a number

From this dictionary and the dictionary for individual constants we see that each of the following sentences of English translates into the formula beside it.

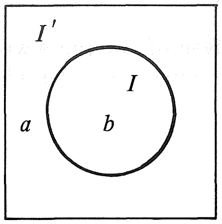
Alan is a farmer	Fa
Betty is happy	Hb
Anne is kind	Kg
The Chancellor is not a miser.	$\sim Mc$
If Alan is kind then Anne is happy.	$Ka \supset Hg$

In our dictionary we have let I stand for *being an insect*. Let us now have a look at some two-item worlds. We will let the two items be a and b. We will use diagrams to represent the two-item worlds. In the diagrams, set out below, we use I' to mean *non-I*. Inside the circle are all the insects, outside the circle are all the non-insects.

(A)



(B)



(C)

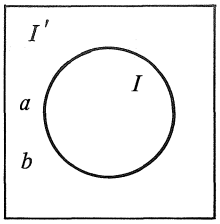


Diagram (A) represents a world in which everything is an insect. Notice how I not only stands for *being an insect*, but also for *the set of insects*. In (A) everything is a member of the set of insects. I' not only stands for *being a non-insect* but also for the *set of non-insects*. I' is an empty set. Together I and I' divide the domain of items.

What is even more interesting is that in (A) it is clear that

Ia is true
Ib is true
$\sim Ia$ is false
$\sim Ib$ is false

Ia is true just because *a* is a member of the set *I* (insects), and so on.

In Diagram (B) we have a world in which at least one thing is an insect and at least one thing is not. Again we see that *I* and I' are sets, and together they divide the world.

In (B) we see that:

Ia is false *Ib* is true

Ia is false because *a* is not a member of the set *I* (insects). Furthermore, since *a* is not a member of the set *I* it must be a member of the set I'.

In Diagram (C) we can see that

Ia is false *Ib* is false

There are no *I*s (insects) in (C). *I* is an empty set, even though the world is non-empty.

Of course, there is a fourth world. Can you work out a Diagram (D)?

Predicate letters can be seen as standing for sub-sets of items in a world, including, of course, either the empty sub-set or the whole world.

A world may always be divided into two sub-sets for each predicate letter. One of the sub-sets goes with the letter, say *I*, and the other subset goes with the *complement of the letter, non-I* or I'. Any given item in a world will have to be in either the *I* or I' subset of that world.

All of this can now be drawn together:

> Individual constants are used to stand for items in a world. Each individual constant stands for one and only one item.
>
> Predicate letters are used to stand for properties which items in a world have or do not have.

Definitions:

> **For any predicate letter, *F*, and any individual constant, *a*, *Fa* is true in a world iff $a \in F$ in that world**
>
> or
>
> ***Fa* is true in a world iff *a* has the property *F* in that world.**

A detailed description can be given of any finite world by setting out a table. The properties are listed across the top, the items down the side.

Properties

		F	*G*	*H*	...
	a				
Items	*b*				
	c				
	d				
	⋮				

Say we have a two item world, the items being Alan and Betty. And we are also interested in four properties:

Fx = x is a farmer a = Alan
Gx = x is a grocer b = Betty

Hx = x is happy
Mx = x is a miser

We can describe one two-item world by the table:

	F	G	H	M
a	1	0	1	0
b	0	1	1	0

In this world it is true of Alan that he is a farmer, and is happy. It is false of him that he is a grocer or a miser. So in our world

Fa is true Ga is false
Ha is true Ma is false

It is true of Betty that she is a grocer and is happy. It is false of her that she is a farmer or a miser. So

Gb is true Fb is false
Hb is true Mb is false

It must follow from the false propositions that, in our two item world

$\sim Ga$ is true $\sim Fb$ is true
$\sim Ma$ is true $\sim Mb$ is true

Furthermore:

$Fa \ \& \ Gb$ is true,
$\sim Ma \ \& \sim Mb$ is true
$Fa \lor Ga$ is true, and so on.

Since there are 8 spaces in the table, each of which may be filled in in 2 different ways (with a 1 or 0), there are 2^8 or 256 different ways of filling in a table of this size.

We finish this section with a note about predicate letters. Predicate letters can stand for whatever property we specify in a dictionary. So, in the context of such a specification the predicate letters are *predicate constants.* But, once we have symbolized a proposition, using the dictionary, the symbols can be seen as displaying the form of that proposition. For example, we might use the dictionaries above to symbolize

Alan is not a farmer (4)

as

$\sim Fa$ (5)

We can now see from (5) that (4) has a certain form.

The form would be the same were the dictionary to be

Fx = x is a frog
a = Albert

and the proposition symbolized was

Albert is not a frog (6)

(5) shows us that the form of (6) and (4) is the same. Since we can vary the dictionary for F, and any predicate letter, predicate letters are sometimes called *predicate variables.* So, MQL is unlike PL. In PL we distinguished propositional variables and constants by using different symbols. In MQL we use the same symbols for both. This does not create any real problems in practice.

EXERCISE 10.4

1. Translate the following into English, using the dictionary supplied, and work out their truth value in the world described by the table set out.

a = Alan
b = Bert
c = Carol
d = Dana

Fx = x is a farmer
Gx = x is a grocer
Sx = x is a student
Tx = x tries hard
Bx = x is a brilliant student

(a) $Fa \,\&\, Sc$
(b) $\sim Gb \,\&\, Tb$
(c) $Bd \supset Sd$
(d) $\sim Bc \,\&\, Sc \,\&\, Tc$
(e) $(Fd \vee Gd) \supset Td$
(f) $Gd \equiv Sb$
(g) $Fa \not\equiv Gc$
(h) $(Bb \,\&\, Bd) \,\&\, (Tb \,\&\, \sim Td)$
(i) $(Ba \supset Sa) \,\&\, (Bb \supset Sb) \,\&\, (Bc \supset Sc) \,\&\, (Bd \supset Sd)$
(j) $Fa \vee Fb \vee Fc \vee Fd$

	F	*G*	*S*	*T*	*B*
a	1	0	0	1	0
b	0	0	1	1	1
c	0	0	1	1	0
d	0	1	1	0	1

2. Fill in the table set out below, using the dictionary in Question 1, so as to make the following true.

(a) Alan is neither a farmer nor a student
(b) Bert is a brilliant student, and does not try hard.
(c) Carol is not a student, but Dana is a farmer
(d) Dana is either a grocer or a farmer but not both.
(e) If Dana is a farmer then she is not a student.
(f) All brilliant students are students.
(g) Every non-student tries hard.
(h) If Carol is a farmer then Dana is a grocer.
(i) Students are neither farmers nor grocers
(j) Everyone other than students is either a farmer or a grocer.

	F	*G*	*S*	*T*	*B*
a					
b					
c					
d					

10.5 QUANTIFIERS AND FINITE WORLDS

Let us look again at the two item world described by the table:

(E)

	F	*G*	*H*	*M*
a	1	0	1	0
b	0	1	1	0

Fx = x is a farmer
Gx = x is a grocer
Hx = x is happy
Mx = x is a miser

It is clear that both a and b are happy. So (1) is true

$$Ha \,\&\, Hb \tag{1}$$

So, *in this world every thing is happy.*

It is also clear that neither a nor b are misers. So (2) is true

$$\sim Ma \,\&\, \sim Mb \tag{2}$$

So, *in this world nothing is a miser, every thing is a non-miser.*

Now look at the formula

$(\forall x)\ Hx$ (3)

The $\forall$ is a *universal quantifier*, and $(\forall x)$ is read as *Every item, x, is such that.* (3) is read as:

Every item, x, is such that it, x, is happy.

This clearly means:

Every thing is happy. (4)

So (3) is read as (4).

In *any* two item world, where the items are a and b, (3) will be equivalent to (1).

$(\forall x)\ Hx \quad \equiv \quad Ha\ \&\ Hb$ (5)

If everyone is happy then both a and b are happy, and if both a and b are happy then everyone is happy. The right hand expression in (5) is known as *the expansion of* $(\forall x)Hx$ *for a, b.*

In our particular world, E, we see that (3) is true, because (1) is true.

Now consider

$(\forall x) \sim Mx$ (6)

This is read as

Every item, x, is such that it, x, is non-M.

This is the same as

Everyone is a non-miser (7)

(7) is a universally quantified negation. In *any* two item world, where the items are a and b, (6) will be equivalent to (2).

$(\forall x) \sim Mx \quad \equiv \quad \sim Ma\ \&\ \sim Mb$

If everyone is a non-miser then both a and b are non-misers, and if both a and b are non-misers then everyone is a non-miser. Once again, $\sim Ma\ \&\ \sim Mb$ is known as *the expansion of* $(\forall x) \sim Mx$ *for a, b.* You will see that both expansions of universally quantified formulae are *conjunctions.* From now on we will shorten "in a world, where the items are a, ..., b" to "in a world a, ..., b". In our particular world, E, since (2) is true so is (6).

Consider (8), (9) and (10)

$Ga\ \&\ Gb$ (8)
$(\forall x)\ Gx$ (9)
Everyone is a grocer (10)

(9) is read as (10), and (9) is equivalent to (8) in any two-item world a, b: (8) is *the expansion of* (9) *for a, b.* Once again the expansion is a conjunction.

In E (8) is false, so, in E (9) will be false.

We now look at some more complex cases.

$(\forall x)(Gx \vee Fx)$ (11)

(11) is read as:

Every item, x, is such that it, x, is either F or G.

This means

Every one is either a farmer or a grocer (12)

(12) is clearly a universally quantified disjunction, as is (11).

The expansion of (11) for *a, b* will be:

$$(Fa \vee Ga) \mathbin{\&} (Fb \vee Gb) \qquad (13)$$

In any two item world *a, b* both (11) and (12) will be equivalent to (14):

a is either a farmer or a grocer *and*
b is either a farmer or a grocer (14)

Look carefully at (14) and compare it with (12). Of course, (13) is the symbolization of (14).

In world E (13) is true, so (11) is true in E.

Note once again how the expansion of a universally quantified formula is a conjunction.

Consider (15):

$$(\forall x)(Fx \supset Hx) \qquad (15)$$

This is read as

Every item, x, is such that if it, x, is F then it is H.

This means

Every item which is *F* is *H*.

or

Every farmer is happy (16)

(16) is a universally quantified conditional, as is (15). The expansion of (15) for *a, b* will be

$$(Fa \supset Ha) \mathbin{\&} (Fb \supset Hb) \qquad (17)$$

Once again the expansion is a conjunction. By this time you should have noticed the following way that expansion works:

Universally quantified formula	Conjunction
negation	Conjunction of negations
disjunction	Conjunction of disjunctions
conditional	Conjunction of conditionals

Go back and check this for yourself. Consider now the formula (16):

$$(\forall x)\,(Fx \mathbin{\&} Hx) \qquad (16)$$

This reads as:

Everything is both a farmer and happy.
or Every thing is a happy farmer.

(16) is a universally-quantified conjunction. So, if our pattern is followed for expansion we should get a conjunction of conjunctions. The expansion of (16) for *a, b* is (17).

$$(Fa \mathbin{\&} Ha) \mathbin{\&} (Fb \mathbin{\&} Hb) \qquad (17)$$

In any two item world, where the items are *a* and *b*, universally quantified formulae are equivalent to their expansions for *a, b*.

The equivalences we have set out above are now reproduced for careful scrutiny:

$$\begin{array}{lcl}
(\forall x)\, Hx & \equiv & Ha \,\&\, Hb \\
(\forall x) \sim Hx & \equiv & \sim Ha \,\&\, \sim Hb \\
(\forall x)(Fx \vee Gx) & \equiv & (Fa \vee Ga) \,\&\, (Fb \vee Gb) \\
(\forall x)(Fx \supset Hx) & \equiv & (Fa \supset Ha) \,\&\, (Fb \supset Hb) \\
(\forall x)(Fx \,\&\, Hx) & \equiv & (Fa \,\&\, Ha) \,\&\, (Fb \,\&\, Hb)
\end{array}$$

If you look carefully at each right hand expression you will see that each conjunct is closely related to the *scope* of the universal quantification in the left hand expression.

In each case the universal quantification is *with respect to x*. A conjunct is formed by taking the scope and then replacing each x with one individual constant, then we get another conjunct by replacing each x with another individual constant, and so on.

If we had a three item world to consider, we would need to work out the expansions of the universally-quantified formulae for three items – say they were a, b, c. Try it out for yourself before looking at the list below:

$$\begin{array}{lcl}
(\forall x) Hx & \equiv & Ha \,\&\, Hb \,\&\, Hc \\
(\forall x) \sim Hx & \equiv & \sim Ha \,\&\, \sim Hb \,\&\, \sim Hc \\
(\forall x)(Fx \vee Gx) & \equiv & (Fa \vee Ga) \,\&\, (Fb \vee Gb) \,\&\, (Fc \vee Gc) \\
(\forall x)(Fx \supset Hx) & \equiv & (Fa \supset Ha) \,\&\, (Fb \supset Hb\ ) \,\&\, (Fc \supset Hc) \\
(\forall x)(Fx \,\&\, Hx) & \equiv & (Fa \,\&\, Ha) \,\&\, (Fb \,\&\, Hb) \,\&\, (Fc \,\&\, Hc)
\end{array}$$

If the formula has a universal quantification with respect to y then we do the same except that it's y that gets replaced. For example, we expand for a, b, c.

$$(\forall y)(Fy \supset \sim Hy) \quad \equiv \quad (Fa \supset \sim Ha) \,\&\, (Fb \supset \sim Hb) \,\&\, (Fc \supset \sim Hc)$$

The same applies to whatever the letter in the quantification.

Each of the conjuncts in such an expansion is known as an *itemization of the scope of the quantification.* In particular, we have one *itemization to a,* one *to b* and one *to c.*

EXERCISE 10.5A

1. Given the following dictionary symbolize the propositions below.

Mx = x is mental
Tx = x is a thought
Px = x is physical
Sx = x is in space

(a) Every thing is physical.
(b) Every thing is mental.
(c) Every thing is either mental or physical.
(d) Every thought is mental.
(e) Every thing physical is in space.
(f) Every thing is non-physical.
(g) Every thing mental is non-physical.
(h) Every thing is both physical and in space.
(i) Thoughts are not physical.
(j) Every thing in space is physical.

2. What is the expansion of each of the following for a, b? Also, work out the truth value for each formula in world (i) and in world (ii) (by working out the values of the expansions).

   (a) $(\forall x)Fx$
   (b) $(\forall x) \sim Gx$
   (c) $(\forall x)(Fx \supset Gx)$
   (d) $(\forall x)(Fx \supset \sim Gx)$
   (e) $(\forall x)(Fx \vee Gx)$
   (f) $(\forall x) \sim (Fx \vee Gx)$
   (g) $(\forall x)(Fx \,\&\, \sim Gx)$

(i)

	F	*G*
a	1	0
b	1	0

(ii)

	F	*G*
a	1	1
b	0	1

3. What is the expansion of each of the following for the items listed beside it?

(a) $(\forall x)\, Sx$	*a, b*
(b) $(\forall x) \sim Sx$	*a, b, c*
(c) $(\forall x)(Fx \supset Gx)$	*a, b, c*
(d) $(\forall y)(Fy \supset Gy)$	*a, b, c*
(e) $(\forall y)\, Ty$	*a, b, c, d*

4. What (expansion) are each of the following equivalent to in world (i) and in world (ii)? What is their truth-value in (i) and in (ii)?

   (a) $(\forall x)(Fx \vee Gx)$
   (b) $(\forall x)(Fx \supset Gx)$
   (c) $(\forall x)(Fx \supset \sim Gx)$
   (d) $(\forall x) \sim Fx$
   (e) $(\forall x)(Fx \equiv Gx)$

(i)

	F	*G*
c	1	0
d	0	1

(ii)

	F	*G*
b	0	0
e	1	1
f	0	0

(E)

	F	*G*	*H*	*M*
a	1	0	1	0
b	0	1	1	0

In the world *a, b* set out in table E we see that (18) is true.

$$Fa \vee Fb \tag{18}$$

Now look at (19).

$$(\exists x)\, Fx \tag{19}$$

The $\exists$ is the *existential quantifier.* $(\exists x)$ is read as *At least one item, x, is such that.* So we read (19) as:

At least one item, x, is such that it, x, is F.

or At least one thing is a farmer. (20)

Now (20) is true in table E precisely because (18) is true. (18) is the expansion of (19) for *a, b*.

In the same world it is clear that (21) is true

$$(Fa \,\&\, Ha) \vee (Fb \,\&\, Hb) \tag{21}$$

Now consider (22)

$$(\exists x)\, (Fx \,\&\, Hx) \tag{22}$$

which reads as:

At least one item is both a farmer and happy
or At least one farmer is happy. (23)

(23) is true in table E precisely because (21) is true. In world E (23) means

either *a* is a happy farmer or *b* is a happy farmer. (24)

(21) is the expansion of (22) for *a, b*.

Existentially quantified formulae expand to disjunctions. (19) expanded to a disjunction of SP atomic wffs. (22) expanded to a disjunction of conjunctions. Each disjunct, in each case, is an *itemization of the scope of the existential quantification.*

We now list expansions for (19) and (22) with others.

$$\begin{array}{lcl}
(\exists x)\, Fx & \equiv & Fa \vee Fb \\
(\exists x) \sim Gx & \equiv & \sim Ga \vee \sim Gb \\
(\exists x)(Fx \,\&\, Hx) & \equiv & (Fa \,\&\, Ha) \vee (Fb \,\&\, Hb) \\
(\exists x)(Fx \supset Hx) & \equiv & (Fa \supset Ha) \vee (Fb \supset Hb) \\
(\exists x)(Fx \vee Gx) & \equiv & (Fa \vee Ga) \vee (Fb \vee Gb)
\end{array}$$

Here also we have a clear pattern:

Existentially quantified formula	Disjunction
negation	disjunction of negations
conjunction	disjunction of conjunctions
conditional	disjunction of conditionals
disjunction	disjunction of disjunctions

We now set out one example of an expansion, but for a four item world, *a, b, c, d.*

$$(\exists x) \sim Hx \quad \equiv \quad \sim Ha \vee \sim Hb \vee \sim Hc \vee \sim Hd$$

The following points should be noted for finite worlds of more than one item:

Universally Quantified formulae expand to conjunctions

Existentially Quantified formulae expand to disjunctions

What happens, then, for one item worlds? The answer is of stunning simplicity, and is best set out at first by examples. Assume that the one item world has *a* as its one item.

$$\begin{array}{lcl}
(\forall x)\, Fx & \equiv & Fa \\
(\exists x)\, Fx & \equiv & Fa \\
(\forall x)(Fx \,\&\, Hx) & \equiv & Fa \,\&\, Ha \\
(\exists x)(Fx \,\&\, Hx) & \equiv & Fa \,\&\, Ha \\
(\forall x)(Fx \supset Gx) & \equiv & Fa \supset Ga \\
(\exists x)(Fx \supset Gx) & \equiv & Fa \supset Ga \\
(\forall x) \sim Hx & \equiv & \sim Ha \\
(\exists x) \sim Hx & \equiv & \sim Ha
\end{array}$$

The expansion, in each case, is simply the one and only possible itemization of the scope of the quantification.

We can now set out the truth conditions for quantifications in finite worlds:

$(\forall x)\ \alpha$ is true in an *n* membered world iff the conjunction of all *n* itemizations of α is true.

$(\exists x)\ \alpha$ is true in an n membered world iff the disjunction of all n itemizations of α is true.

We have:

World	$(\forall x)\ \Phi x$	$(\exists x)\ \Phi x$
a	Φa	Φa
$a\ \ b$	$\Phi a\ \&\ \Phi b$	$\Phi a \vee \Phi b$
$a\ \ b\ \ c$	$\Phi a\ \&\ \Phi b\ \&\ \Phi c$	$\Phi a \vee \Phi b \vee \Phi c$
$a\ \ b\ \ldots$	$\Phi a\ \&\ \Phi b\ \&\ \ldots$	$\Phi a \vee \Phi b \vee \ldots$
⋮	⋮	⋮

where Φx is the scope of the quantification. (We will see later, § 11.2, that x must occur "free" in Φx).

EXERCISE 10.5B

1. Given the following dictionary symbolize the propositions below.

Mx = x is mental

Px = x is physical

Tx = x is a thought

Sx = x is in space

(a) Some thing is a thought.
(b) Something is not in space
(c) Something is either mental or physical
(d) Something in space is physical
(e) Something in space is not physical

2. What is the expansion of each of the following for a, b? Also work out the truth-value of each formula in (i) and in (ii).

(a) $(\exists x) \sim Fx$
(b) $(\exists x)(Fx\ \&\ Gx)$
(c) $(\exists x)(Fx \vee Gx)$
(d) $(\exists x)(Fx \supset Gx)$
(e) $(\exists x)(Fx\ \&\ \sim Gx)$
(f) $(\exists x) \sim(Fx \vee Gx)$
(g) $(\exists x) \sim(Fx \supset \sim Gx)$

(i)

	F	G
a	1	0
b	1	0

(ii)

	F	G
a	1	1
b	0	1

3. What is the expansion of each of the following for the items listed beside it?

(a) $(\exists y) \sim Sy$	a, b, c
(b) $(\exists z)(Sz\ \&\ Tz)$	a, b, c
(c) $(\exists y)(Sy\ \&\ \sim Ty)$	a, b
(d) $(\exists y) \sim(Sy \supset \sim Ty)$	a, b, c
(e) $(\exists x)(Sx\ \&\ Fx)$	a, b, c, d

4. What are each of the following equivalent to in world (i) and in world (ii)? What truth-value do they have in (i) and (ii)?

(a) $(\exists x)\ Fx$
(b) $(\exists x) \sim Gx$
(c) $(\exists y)(Fy \vee Gy)$
(d) $(\exists z)(Fz \not\equiv Gz)$
(e) $(\exists x)(Fx\ \&\ \sim Gx)$

(i)

	F	G
a	1	0
b	0	1
c	1	0

(ii)

	F	G
a	0	0
e	0	1

We now turn to some more complicated cases. So far we have considered only those cases where there is one quantifier and it is the main operator in the formula. Consider formula (25):

$$\sim(\forall x)Fx \tag{25}$$

This formula can be expanded for a, b by replacing the quantified sub-formula with its expansion. This gives (26), with the tilde unaltered:

$$\sim(Fa \;\&\; Fb) \tag{26}$$

Similarly, we expand (27) for a, b to get (28)

$$(\forall x)\, Sx \lor (\forall x)\, Mx \tag{27}$$

$$(Sa \;\&\; Sb) \lor (Ma \;\&\; Mb) \tag{28}$$

The expansion of (29) for a, b, c gives (30)

$$\sim(\exists x) \sim Fx \tag{29}$$

$$\sim(\sim Fa \lor \sim Fb \lor \sim Fc) \tag{30}$$

Finally, for this section, we consider (31), which contains an individual constant, a within the scope of the quantification.

$$(\forall x)(Fx \supset Ga) \tag{31}$$

Two things are to be noted about such formulae. First, the individual constant does not change in any itemization. Second, one of the items for which this formula is expanded must be a. The expansion of (31) for a, b, c is (32).

$$(Fa \supset Ga) \;\&\; (Fb \supset Ga) \;\&\; (Fc \supset Ga) \tag{32}$$

Only the x is replaced.

If we were asked to expand (31) for b, c, then one or other of the items b and c must also be named "a" as well. The expansion of (31) for b, c is (33))

$$(Fb \supset Ga) \;\&\; (Fc \supset Ga) \tag{33}$$

If we are not told whether a is b or a is c then we might not be able to determine the value of (33), and not of (31). Consider the world:

	F	G
a	0	1
b	1	0

If a is b then (33) is true.
If a is c then (33) is false.

So, to avoid confusion, we generally make clear what is what when individual constants appear as in (31).

So far we have always spoken about *expanding a formula for some constant(s)*. In some cases we will use *eliminating the quantifiers in a formula for some constant(s)* to mean the same thing.

One final comment: In this section we have not considered any formula where one quantification is in the scope of another. We will leave such formulae until later. We will not consider them in the next section either. Nor have we considered formulae containing propositional variables or propositional constants. These we also leave to a later section.

EXERCISE 10.5C

1. Given the following dictionary symbolize the propositions below:

a = Antares b = the number two
Px = x is physical Ix = x is an idea
Mx = x is mental Sx = x is in space

(a) Antares is in space.
(b) The number two is not in space.
(c) If Antares is in space then something is in space.
(d) If the number two is not in space then not every thing is in space.
(e) Every idea is mental.
(f) Something is mental and something is not.
(g) If Antares is physical then Antares is not an idea.
(h) Not every idea is mental.
(i) Not every idea is non-physical.
(j) Either the number two is in space or not every thing is in space.

2. Eliminate quantifiers in the following for the listed items.

(a)	$\sim(\exists x)\sim Fx$	a, b
(b)	$\sim(\forall x)\sim Fx$	a, b
(c)	$\sim(\exists x)\sim Fx$	a
(d)	$\sim(\forall x)\sim Fx$	a
(e)	$(\exists x)\, Gx \supset (\forall x)\, Gx$	b
(f)	$(\exists x)\, Gx \supset (\forall x)\, Gx$	b, c
(g)	$(\forall x)\sim Gx \supset (\exists x)\sim Gx$	a, b
(h)	$(\forall x)(Gx \supset Fx) \supset (\forall y)(\sim Fy \supset \sim Gy)$	a, b
(i)	$(\forall x)(Gx \supset Fx) \supset (\exists y)(Gy \,\&\, Fy)$	a, b
(j)	$(\exists x)(Fx \,\&\, Ga)$	a, b, c
(k)	$(\exists x)\, Fx \,\&\, Ga$	a, b, c
(l)	$(\forall x)(Fx \supset Ga)$	a, b
(m)	$(\forall x)\, Fx \supset Ga$	a, b
(n)	$\sim(\,(\forall x)\, Fx \supset (\exists x)\, Gx\,)$	a, b
(o)	$\sim(\,(\exists x)(Fx \,\&\, Gx) \,\&\, (\forall x)\sim Fx)$	b, c

*3. Eliminate quantifiers in the following for the two items: b, c

(a) $(\exists x)(Fx \vee Ga)$
(b) $(\forall x)(Fx \vee Ga)$
(c) $(\exists x)(Fx \vee Gc)$
(d) $Gb \,\&\, (\exists x)(Fx \vee Fa)$
(e) $(\forall x)(\sim Fx \supset Fa)$

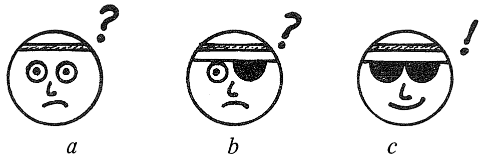
Some of the formulae above will be true in the world set out below, even in the absence of complete information, and some will be false. Some will be left undetermined. Which are which? Explain your answers.

	F	G
b	1	0
c	0	1

Puzzle 10

A villain known as "The Puzzler" has captured three students, *a, b* and *c*, who are well known for their brilliance at logic. Although *a* has normal vision, *b* has only one eye and *c* is blind in both eyes. The Puzzler tells his captives that he has five headbands, three blue and two green, and he then places one of these on each of their heads in a way such that none of the students can see the colour of their own headband. He then tells them that if any of them can correctly guess their own colour he will set them all free; but if any guess the wrong colour they will stay imprisoned for good. *a* looks at his fellow captives and admits: "I don't know my colour". Then *b* also looks about him and confesses: "I don't know my colour either". At this point *c* smiles and correctly gives his own colour. Somewhat impressed, the Puzzler releases them and remarks to *c*: "Your logical eyes see very well".

What was the colour of *c*'s headband, and how did he deduce it?



10.6 SUMMARY

Monadic Quantification Theory (MQT) subsumes PC, and also provides a means of dealing with simple notions of *quantity* (all, some, none) as well as *properties* of individual *items*. Its language is MQL.

Singular reference (i.e. reference to a single item) is made by means of *singular terms*, which may be *proper names* (e.g., "Alan"), *singular pronouns* (e.g., "he") or *definite descriptions* (e.g., "the Pope"). Items may be real or fictional. With singular reference, the items are said to be logical subjects, and the properties or relations ascribed to them are logical *predicates*. Phrases which express quantity are called *quantifiers*. "All" and "every" are *universal* quantifiers, while "some" and "at least one" are particular, or *existential*, quantifiers. When quantifying over discrete items we will read "some" as "at least one" unless it is obvious that further information is intended (e.g., "more than one", "not all").

In the following English language *square of opposition*, four types of proposition are represented: **A** (universal affirmative); **E** (universal negative); **I** (particular affirmative); **O** (particular negative). Each proposition is *contradictory* to (and hence equivalent to the negation of) its diagonally opposite partner.

Every *A* is *B*	**A**	**E**	No *A* is *B*
Some *A* is *B*	**I**	**O**	Some *A* is not *B*

MQL adds the following to PL:

individual constants (*IC*s)	$a, b, c, \ldots$
individual variables (*IV*s)	$x, y, z, \ldots$
predicate constants	$A, B, C, \ldots$
predicate variables	$F, G, H, \ldots$
quantifiers	$\forall, \exists$

ICs and IVs are collectively called *individual letters*, and predicate constants and predicate variables are *predicate letters*. The term "individual" should be read as *"item"*, and within the context of MQT the term "predicate" should be read as *"property"*. Let ν denote any IV, s denote any individual letter, and S denote any predicate letter. Then the formation rules of MQL are as for PL with the following additions:

B2M:	Anything of the form Ss is a wff	e.g., Fa
R$\forall$:	If α is a wff so is $(\forall\nu)\alpha$	e.g., $(\forall x)\ Fx$
R$\exists$:	If α is a wff so is $(\exists\nu)\alpha$	e.g., $(\exists y)\ (Fy\ \&\ {\sim}(\forall x)\ Fx)$

In $(\forall\nu)\alpha$, α is said to be the *scope* of the *quantification* $(\forall\nu)$. Similarly for $(\exists\nu)\alpha$. So, like $\sim$, quantifications have minimum scope.

We stipulate that each *world* contains at least one item: the set of items in a world is the *domain* of that world. *Set diagrams* may be constructed for any given world: on such diagrams we use F to denote the set of items with property F, and F' to denote the *complement* of F (i.e. the set of items which do not have the property F). Worlds may also be described by *tables*: here 1 or 0 is entered in a cell to indicate that the item on that row has or hasn't the property on that column.

$\forall$ is a *universal quantifier*. The quantification $(\forall x)$ may be read as "For all x", or more strictly as "For each and every item x, taken one at a time". For instance, if "Fx" denotes "x is a frog" then $(\forall x)Fx$ means "Every thing is a frog". Here "thing" means an item in the world.

$\exists$ is an *existential quantifier*. The quantification $(\exists x)Fx$ may be read as "Some item x is such that" where "some" has the minimal reading "at least one". For instance, if "Fx" denotes "x is a frog" then $(\exists x)Fx$ means "Some thing is a frog".

Consider a formula ϕx where no occurrence of x lies in the scope of a quantification. Then ϕa, the result of substituting a for x, is called an *itemization* of ϕx with respect to a. For example, if $\phi x = Fx \supset Gx$ then $\phi a = Fa \supset Ga$. In any finite world, a quantified formula is equivalent to its *expansion* in that world. Universally quantified formulae expand to *conjunctions* of their itemizations, and existentially quantified formulae expand to *disjunctions* of their itemizations.

Domain	$(\forall x)\ \phi x$	$(\exists x)\ \phi x$
a	ϕa	ϕa
a, b	$\phi a\ \&\ \phi b$	$\phi a \vee \phi b$
a, b, c	$\phi a\ \&\ \phi b\ \&\ \phi c$	$\phi a \vee \phi b \vee \phi c$
$\vdots$	$\vdots$	$\vdots$

For example, in any world of two items a and b we have:

$$(\forall x)\ (Fx \supset Gx) \Leftrightarrow (Fa \supset Ga)\ \&\ (Fb \supset Gb)$$
$$(\exists y)\ (Fy \equiv Gy) \Leftrightarrow (Fa \equiv Ga) \vee (Fb \equiv Gb)$$

11

Necessary Truth and Validity

11.1 TRANSLATION

It should, by now, be fairly clear how the symbols of MQL are to be used for translation. This section will add a few hints and some cautionary notes.

Care must be taken when symbolizing the logical predicates in propositions. For example (1) is equivalent (2), so (1) can be translated as (3).

John is a two metre tall man. (1)
John is a man and John is two metres tall. (2)

Let j = John
Tx = x is two metres tall
Mx = x is a man

$Tj \;\&\; Mj$ (3)

In Staines Arrows terms. (1) $\leftrightarrow$ (2)
(1) $\leftrightarrow$ (3)

This is to be contrasted with the relationship between (4) and (5), which are not equivalent.

Namu is a small whale. (4)
Namu is a whale and Namu is small. (5)

(4) should be translated as (6), not as (7).

Let n = Namu
Sx = x is a small whale
Wx = x is a whale
Ax = x is small

Sn (6)
$Wn \;\&\; An$ (7)

If we also have formula (8), the relationships are that (4) is equivalent to (6), and (4) implies (8). (4) does not imply (7).

Wn (8)
(4) $\leftrightarrow$ (6)
(4) $\rightarrow$ (8)

We have already seen that "some" *implies* the existential quantification in MQT. A similar procedure applies to the following:

$$\left.\begin{array}{ll} \text{a couple} & (\text{a couple of things are } F) \\ \text{a few} & (\text{a few things are } F) \\ \text{many} & (\text{many things are } F) \\ \text{most} & (\text{most things are } F) \end{array}\right\} \longrightarrow \quad (\exists x)\, Fx$$

But note that the converse implication fails.

It is often useful to regiment ordinary language before symbolizing. Many quantified propositions can be re-expressed in sentences of *A, E, I* or *O* form.

A	*Every ... is ...*	$(\forall \nu)(\ldots \supset \ldots \quad)$
I	*Some ... is ...*	$(\exists \nu)(\ldots \;\&\; \ldots \quad)$
E	*Not even one ... is ...*	$\sim (\exists \nu)(\ldots \;\&\; \ldots \quad)$
O	*Some ... is not ...*	$(\exists \nu)(\ldots \;\&\; \sim \ldots \;)$

The **A** and **I** forms are very useful. It is then a simple matter to symbolize. For example, when asked to symbolize (9) we first put it into **A** form (10).

Hollow spheres are all green. (9)
Every sphere which is hollow *is* green. (10)

The phrases "which is", "who is", and "that is" often indicate conjunction. So, the antecedent of this universally-quantified conditional, (10), is a conjunction. With the dictionary below we get (11)

Let $Sx = x$ is a sphere
$Hx = x$ is hollow
$Gx = x$ is green.

$$(\forall x)[(Sx \;\&\; Hx) \supset Gx] \tag{11}$$

In like manner we first put (12) into (13).

Anyone who has fun is either a blonde or a red-head. (12)
Every person who has fun *is* either a blonde or a red-head. (13)

The antecedent of this universally quantified conditional is a conjunction and the consequent is a disjunction. With the dictionary below we get (14).

Let $Px = x$ is a person $Fx = x$ has fun
$Bx = x$ is a blonde $Rx = x$ is a red-head

$$(\forall x)[(Px \;\&\; Fx) \supset (\, Bx \vee Rx)] \tag{14}$$

In like fashion, we regiment (15) to **I** form to get (16). Using the dictionary above we translate to (17).

Some hollow spheres are green (15)
Some sphere which is hollow *is* green (16)

$$(\exists x)[(Sx \;\&\; Hx) \;\&\; Gx] \tag{17}$$

It is often best to regiment the **E** propositions into the form of a negated **I**. So (18) becomes (19), which then translates as (20).

No hollow sphere is green (18)
Not even one sphere which is hollow *is* green. (19)

$$\sim(\exists x)[(Sx \;\&\; Hx\,) \;\&\; Gx] \tag{20}$$

Many quantification phrases of English contain within them some property. This is clear with "everybody" "somebody", "everyone" and "someone". The property of being a person is built into the quantification. Thus we see that (21) and (22) are equivalent to (23):

Somebody is tall (21)
Someone is tall (22)
Some person is tall (23)

Similarly (24) and (25) are equivalent to (26)

Everybody is tall (24)
Everyone is tall (25)
Every person is tall (26)

The same can be said of "thing", "time" and "place" in "something", "sometime", and "somwhere".

Such properties need to be taken into account when translating into MQL. We mostly ignore the term "thing". But the others are usually taken into account in one way or another.

EXERCISE 11.1

In questions 1–3 translate into MQL using only the dictionary provided.

1. Ox = x is over two metres tall
   Px = x is a person
   Ix = x is intelligent
   Mx = x is a machine
   Bx = x has blue eyes

   (a) Some things are over two metres tall.
   (b) Some things are not over two metres tall.
   (c) Not everybody is over two metres tall.
   (d) Some one is over two metres tall.
   (e) Some one with blue eyes is over two metres tall.
   (f) Everyone has blue eyes.
   (g) Everyone has blue eyes and is over two metres tall.
   (h) Everyone who has blue eyes is over two metres tall.
   (i) No one who has blue eyes is over two metres tall.
   (j) No one who is over two metres tall is intelligent.
   (k) Some one who is intelligent is either blue eyed or over two metres tall.
   (l) Some machine is intelligent.
   (m) Some machine which is over two metres tall is not intelligent.
   (n) At least one intelligent machine is a person.
   (o) Machines do not have blue eyes.

2. Fx = x is fruit
   Dx = x is delicious
   Ax = x is an apple
   Vx = x is a vegetable
   Nx = x is nutritious
   Ox = x is an orange
   Gx = x is good to eat
   Jx = x is juicy

   (a) Some fruit is delicious
   (b) Apples are all delicious
   (c) Some vegetables are not delicious
   (d) Some vegetables which are not delicious are, nevertheless, nutritious.
   (e) Apples are not vegetables.
   (f) Oranges are delicious and nutritious
   (g) Some vegetables are not nutritious.

(h) Not everything which is good to eat is nutritious.
(i) If an apple is delicious then it is good to eat.
(j) If fruit is juicy then it is not a vegetable.
(k) Apples and oranges are delicious and nutritious.
(l) All vegetables which are not good to eat are nutritious.
(m) All fruits other than apples and oranges are juicy, delicious, and nutritious.
(n) If something is delicious then it's good to eat.
(o) Some fruit is juicy and some is not.

3. Px = x is a proposition
Tx = x is true
Fx = x is false
Cx = x is contingently true
Wx = x tells us about the world.
Nx = x is necessarily true.

(a) Propositions are either true or false.
(b) A thing is true if and only if it's not false.
(c) No proposition is both true and false.
(d) Some propositions are neither true nor false.
(e) Some propositions are true, but not necessarily true.
(f) Propositions which are true, but not necessarily so, are contingently true.
(g) Only true propositions tell us about the world.
(h) None but the true propositions tell us about the world.
(i) Necessarily true propositions are not contingently true.
(j) If a proposition is not necessarily true then it is either contingently true or false.

11.2 FREE AND BOUND OCCURRENCES OF VARIABLES

Before continuing we must deal with an important point in syntax. It concerns the small letters at the end of the alphabet. They are called *individual variables*, because when a formula is expanded the individual variables are replaced by individual constants. Sometimes the individual variables are called "variables of quantification".

Individual variables can occur in the wffs of MQL outside the scope of any quantification. In the following examples we have underlined such cases:

$$F\underline{x}, \qquad F\underline{x} \mathbin{\&} (\exists x)\, Gx, \qquad (\forall y)\, Gy \supset G\underline{y}$$

Each underlined individual variable *occurs free* where underlined. All the others *occur bound.* But furthermore, if an individual variable is within the scope of a quantification, but not a quantification with respect to it, it still occurs free. In the following examples we have underlined the free occurrences.

$$(\exists x)F\underline{y} \qquad (\forall y)(Fy \supset G\underline{x}) \qquad (\exists y)(\exists z)(Fy \mathbin{\&} Gz \mathbin{\&} H\underline{x})$$

The individual variables inside the quantifications occur bound.

Definition: Any occurrence of an individual variable v either in, or within the scope of, a quantification with respect to v is said to be *bound*. An occurrence which is not bound is said to be *free.*

If an individual variable occurs bound then it will be *bound by a quantifier*. Being *bound by* can be displayed with binding lines. For example:

$$(\forall \underbrace{x)Fx} \qquad (\exists \underbrace{x)(Fx}\underbrace{\ \&\ Gx}) \qquad (\forall \underbrace{y)(Fy} \supset Gx)$$

If we take the scope of a quantification w.r.t. v as a separate wff, then all free occurrences of ν in the separate wff will be bound by the quantifier. We have set out four examples below. On the left is the formula, next is the scope as a separate wff with free occurrences underlined, next is the binding for the left most quantifier and on the right are all the binding lines.

$(\forall x)\, Fx$	$F\underline{x}$	$(\forall \underbrace{x)Fx}$	$(\forall \underbrace{x)Fx}$ $\quad(\forall \underbrace{y)\, Fy}$
$(\forall x)((\forall x)Fx \;\&\; Fx)$	$((\forall x)\, Fx \;\&\; F\underline{x})$	$(\forall \underbrace{x)((\forall x)\, Fx \;\&\; Fx})$	$(\forall \underbrace{x)((\forall \underbrace{x)\, Fx} \;\&\; Fx})$
$(\forall x)(Fx \supset (\forall y)Fy)$	$(F\underline{x} \supset (\forall y)\, Fy)$	$(\forall \underbrace{x)(Fx} \supset (\forall y)\, Fy)$	$(\forall \underbrace{x)(Fx} \supset (\forall \underbrace{y)Fy})$
$(\forall y)(Gy \supset Fx)$	$(G\underline{y} \supset F\underline{x})$	$(\forall \underbrace{y)(Gy} \supset F\underline{x})$	$(\forall \underbrace{y)(Gy} \supset F\underline{x})$

As a formula is constructed in an assembly line the binding which first occurs is the one which persists. Later additions or reduplications of quantifications have no effect. For example:

1. Fx	B2M
2. $(\forall \underbrace{x)Fx}$	1, R $\forall$
3. $(\exists x)(\forall \underbrace{x)Fx}$	2, R $\exists$
4. $(\forall x)(\exists x)(\forall \underbrace{x)Fx}$	3, R $\forall$

When a quantifier binds nothing in its scope it is a *vacuous quantifier*. The quantifiers added in the last two steps of the assembly line above are vacuous.

If a formula has in it any free occurrences of individual variables it is an *open formula*. An open formula can be closed by the placing of quantifications to the left, one for each variable which occurs free. If all the quantifications so placed are universals then we get the *universal closure*. If all are existentials we get the *existential closure*. For example, we set out open formulae to the left and both closures to the right:

Fx	$(\forall x)Fx$ $(\exists x)Fx$
$(Fx \supset Gx)$	$(\forall x)(Fx \supset Gx)$ $(\exists x)(Fx \supset Gx)$
$Fx \;\&\; Gx$	$(\forall x)(Fx \;\&\; Gx)$ $(\exists x)(Fx \;\&\; Gx)$
$(\forall x)(Fx \supset Gy)$	$(\forall y)(\forall x)(Fx \supset Gy)$ $(\exists y)(\forall x)(Fx \supset Gy)$

When the quantifiers are all of one kind it does not matter in what order they are added for closure. So we can universally close $(Fx \supset Gy)$ by either $(\forall x)(\forall y)(Fx \supset Gy)$
or $(\forall y)(\forall x)(Fx \supset Gy)$

Be careful – when closing a formula be sure to add any missing outer parentheses *before* adding quantifications.

We also introduce a useful piece of notation:

$$(Q\nu) \ldots (Q\,\omega)\,\alpha \tag{F}$$

(F) is used to denote any wff with an uninterrupted string of quantifications to the left. There can be one or more quantifications and Q stands for either $\forall$ or $\exists$. The following all have the form of (F).

$(\exists x)Fx \quad , \quad (\forall x)(Fx \supset Gx) \quad , \quad (\forall x)(\exists y)(Fx \mathbin{\&} Gy) \quad , \quad (\exists x)\, p \quad ,$
$(\forall x)(\forall x)(\exists y)(\forall z)(\exists w)(p \supset (Gx \supset Fy))$

Note also that in (F), if α is well-formed strictly in accordance with the formation rules it must be the scope of $(Q\ \omega)$.

EXERCISE 11.2

1. Write down the scope of $(\forall x)$ in the following expressions.
   - (a) $(\forall x)Fx \equiv Gx$
   - (b) $(\forall x)(Fx \vee (\exists y)Gy)$
   - (c) $(\exists y)(\forall x)(Fx \supset (\forall y)Fy)$

2. Write down the scope of $(\exists x)$ in the following expressions.
   - (a) $(\exists x)(Hx \supset p)$
   - (b) $\sim(\exists x)\, Hx \supset p$
   - (c) $(\forall y)(\forall z)[(\exists x)(Lx \mathbin{\&} Lz) \supset (Py \mathbin{\&} Pz)]$

3. Indicate the bound occurrences of all individual variables in the following formulae by linking those occurrences which must refer to the same object.
   - (a) $(\exists x)(Fx \equiv Gx)$
   - (b) $(\exists x)Fx \equiv Gx$
   - (c) $(\exists x)(\forall y)\ \ Fy$
   - (d) $Fx \mathbin{\&} (\exists x)Gx$
   - (e) $(\forall y)[Hy \supset (\forall y)\, Gy]$
   - (f) $(\exists z)[\ \sim Gz \vee (\exists z)Fz]$
   - *(g) $\sim(\forall y)[(\exists x)\, Fx \mathbin{\&} Gy] \supset (\forall x)(\forall y)\, Gy$
   - *(h) $(\exists x)[(Fy \supset Gx) \equiv (\forall x)Fx]$

4. Write out the following wffs and underline every free occurrence of any individual variable.
   - (a) Fx
   - (b) $Fx \mathbin{\&} (\forall x)Gx$
   - (c) $(\forall x)Gx \mathbin{\&} Fx$
   - (d) $(\forall z)((Fx \mathbin{\&} Fy) \supset Gz)$
   - (e) $(p \mathbin{\&} Fx) \supset (\exists x)Fx$
   - (f) $Fx \supset Gy$
   - (g) $(\exists x)((\forall y)Gy \mathbin{\&} (Gx \mathbin{\&} Fy))$

5. Write out both universal and existential closures of all the wffs in 4.

6. Write out the following wffs with the vacuous quantifications underlined.
   - (a) $(\forall y)(\exists x)((\forall x)Fx \supset Gz)$
   - (b) $(\forall z)((\exists z)Fz \mathbin{\&} (\forall z)Fz)$
   - (c) $(\forall z)(\exists y)\, Fx$
   - (d) $(\forall x)\, p$
   - (e) $(\forall x)\, p \supset Fx$

11.3 NECESSARY TRUTH AND EQUIVALENCES IN MQT.

We begin by distinguishing those formulae of MQL which are *MQL-forms*. When there is no dictionary, the formula is an MQL-form. In any MQL-form the predicate letters are predicate variables and the individual constants are pseudo-constants. Some formulae of MQL are sentences of MQL, i.e. they express propositions. Any sentence of MQL must have a dictionary for all its predicate letters, individual constants and propositional constants, and cannot contain any propositional variables (See §2.5.)

The relationship between forms and propositions in MQL is similar to that in PL. (See §4.3.) Some MQL-forms are true in every world. They are *MQT-Necessities.* Some are false in every world. These are *MQT-Contradictions.* The remaining MQL-forms are *MQT-Contingent.* Any proposition which has an MQT-Necessity for one of its forms will be necessarily true and is itself called an MQT-Necessity. Any proposition which has an MQT-Contradiction for one of its forms will be necessarily false and is itself called an MQT-Contradiction. Any proposition whose explicit form in MQL is an MQT-Contingency will be *MQT-Indeterminate*. We now set out the definition of *MQT-Necessity* for forms.

Definition: A form is an MQT-Necessity iff it is true in every world.

For example: consider (1)

$$(\forall x)(Fx \supset Fx) \tag{1}$$

In order to find out if (1) is an MQT-Necessity we could:

First find out if it is true in every one-item world. This is done by testing its expansion for *every* one-item world. We can do this by simply *naming* the one item "*a*". So, all we have to test is (2).

$$Fa \supset Fa \tag{2}$$

If (2) is a tautology then (1) must be true in every one-item world. And so it is.

Second, we find out if (1) is true in every two-item world. This is done by testing its expansion for every two-item world. We can do this by naming the two items in any two-item world: "*a*" and "*b*". So, all we have to test is (3):

$$(Fa \supset Fa\ ) \ \&\ (Fb \supset Fb) \tag{3}$$

Since (3) is a tautology (1) must be true in every two-item world.

Third, fourth, fifth, etc, we could go on. But, it is not possible to expand a formula for an infinite world. So, what happens when we want to find out whether (1) is true in any infinite world?

Fortunately, there are *three vital facts* about MQT which we will use to simplify things. We now set out fact (*A*).

A. *An MQT-form is true in every finite world iff it is true in every infinite world.*

So, we only have to check finite worlds. This gives us the following result:

A form is an MQT-Necessity iff all its expansions (for finite worlds) are tautologies.

Sad to say, this still leaves us with the prospect of checking infinitely many finite worlds. But the other two vital facts limit the amount of checking even more dramatically.

Let the number of predicate letters in a wff, α, be n. In $(\forall x)(Fx \supset Fx)$ n is *one*, since there is only one predicate letter: F. Now we set out the second vital fact (B):

B *If every expansion of an MQT-form up to* 2^n *items is a tautology, then every finite expansion will be a tautology.*

Since 2^n is *two* for (1), and both the one item and two-item expansions of (1) are tautologies, we may conclude that every expansion of (1) will be a tautology. So (1) is an MQT-Necessity.

Fact B gives us a *finite upper limit* on the size of expansions we need to investigate. This is a completely general feature of MQT. So we can reword our definition of MQT-Necessity to build in this limit.

> **If α is an MQL-form and it contains n predicate letters, then α is an MQT-Necessity iff every expansion of α up to 2^n items is a tautology.**

So, to test (4) for MQT-Necessity we need check only the one and two item expansions.

$$\sim(\forall x)\sim Fx \quad \equiv \quad (\exists x)\, Fx \tag{4}$$

They are set out in (5) and (6).

$$\sim\sim Fa \quad \equiv \quad Fa \tag{5}$$

$$\sim(\sim Fa \,\&\, \sim Fb) \quad \equiv \quad (Fa \vee Fb) \tag{6}$$

Since both (5) and (6) are tautologies, (4) is an MQT-Necessity.

To test (7) we need look only at (8) and (9).

$$(\forall x)Fx \quad \equiv \quad (\exists x)\, Fx \tag{7}$$

$$Fa \quad \equiv \quad Fa \tag{8}$$

$$(Fa \,\&\, Fb) \quad \equiv \quad (Fa \vee Fb) \tag{9}$$

(9) is not a tautology, and is false when:

	F
b	1
c	0

So we have a counterexample, and (7) is not an MQT-Necessity. The counterexample is a two-item world in which one item is F, so something is F, but one item is not F, so not every thing is F. (7) reads as (10), and (10) is clearly not a necessary truth.

Every item is F iff at least one item is F (10)

Fact B gives us a general upper limit on expansions, but look at (11).

$$(\forall x)[Fx \supset (Gx \supset Fx)] \tag{11}$$

Because there are two letters we have the prospect of checking the one, two, three and four item expansions. Things could get worse. There is a third fact which pulls the upper limit down even more dramatically for some wffs. We set out fact (C).

C **If** *a form is closed and of the form*
$(Q\nu) \ldots (Q\,\omega)\,\alpha,$
where α *is quantifier free,*
there are no vacuous quantifiers,
and there are **n** *universal quantifiers,*
then *the form is an MQT-Necessity iff every expansion up to* **n** *items or one item, whichever is larger, is a tautology.*

The following formulae match the description set out in the antecedent of fact *C*;

$$(\forall x)(Fx \supset Fx) \;,\; (\exists x)(Fx \,\&\, Gx) \;,\; (\forall x)(\exists y)(Fx \,\&\, Gy)$$
$$(\forall x)(Fx \supset (Gx \supset Fx)),\; (\forall x)(\exists y)(\exists z)(Fx \supset (Gy \supset Fz))$$

The fact is applied in the following way:

Method:
1. Check that the form matches the description.
2. Count the universal quantifiers.
3. (a) If there are no universal quantifiers we check the one-item expansion only.
   (b) If there are one or more we check out the one to n-item expansions.

Example: Is $(\forall x)[Fx \supset (Gx \supset Fx)]$ an MQT-Necessity?
1. It does match.
2. There is *one* universal quantifier.
3. Since $(Fa \supset (Ga \supset Fa))$ is a tautology, $(\forall x)[Fx \supset (Gx \supset Fx)]$ is an MQT-Necessity.

Example: Is $(\forall x)\,Fx \equiv (\exists x)Fx$ an MQT-Necessity?
1. It does not match – so we must use some other method.

We now set out a test for MQT-Contradiction, incorporating the fact *B* upper limit.

If α is an MQL-form and contains n predicate letters, then α is an MQT-Contradiction iff every expansion of α up to 2^n items is a PC-Contradiction.

The expansions of (12) for one and two items are set out in (13) and (14).

$$(\forall x)Fx \equiv (\exists x) \sim Fx \tag{12}$$
$$Fa \equiv \sim Fa \tag{13}$$
$$(Fa \,\&\, Fb) \equiv (\sim Fa \vee \sim Fb) \tag{14}$$

Since both (13) and (14) are contradictions, (12) is an MQT-Contradiction.

Forms which are neither MQT-Necessities nor MQT-Contradictions are MQT-Contingent. For example, (15) expands for one and two items to (16) and (17).

$$(\forall x)Fx \equiv (\exists x)Fx \tag{15}$$
$$Fa \equiv Fa \tag{16}$$
$$(Fa \,\&\, Fb) \equiv (Fa \vee Fb) \tag{17}$$

Since (17) is contingent, (15) is MQT-Contingent. We need only one contingent expansion.

But pause a moment. A proposition may have the form of an MQT-Contingency and still be necessarily true. For example, with the appropriate dictionary (18) has the form of (19).

All bachelors are unmarried (18)

$$(\forall x)(Bx \supset \sim Mx) \tag{19}$$

In the world set out as follows (19) is false.

	B	M
a	1	1

But when B is the property of *being a bachelor*, and M the property of *being married* then such a world is impossible. It is a counter-model but not a counterexample. See § 7.2.

We now turn to a few of the simpler necessary equivalences of MQT. We have already seen that (20) is an MQT-Necessity.

$$\sim(\forall x)\sim Fx \equiv (\exists x)Fx \tag{20}$$

The following, (21) to (23) are all MQT-Necessities. Proof is left to the reader. Note that fact C cannot be used.

$$\sim(\exists x)\sim Fx \equiv (\forall x)Fx \tag{21}$$
$$\sim(\forall x)Fx \equiv (\exists x)\sim Fx \tag{22}$$
$$\sim(\exists x)Fx \equiv (\forall x)\sim Fx \tag{23}$$

All of (20) to (23) are known as Quantifier Negation or QN.

A useful device for relating and contrasting formulae is the *Square of Opposition* which we first met in §10.2 for ordinary language.

1.

$$\begin{tikzcd}
(\forall x)\,Fx \arrow[rr, leftrightarrow, "MQT\text{-}Contraries"] \arrow[d, Rightarrow, "MQT\text{-}Implies"'] \arrow[drr, leftrightarrow] & & (\forall x)\sim Fx \arrow[d, Rightarrow, "MQT\text{-}Implies"] \arrow[dll, leftrightarrow, "MQT\text{-}Contradictories" description] \\
(\exists x)Fx \arrow[rr, leftrightarrow, "MQT\text{-}Sub\text{-}Contraries"'] & & (\exists x)\sim Fx
\end{tikzcd}$$

It is easy to show that the diagonally opposite formulae are contradictories because each is equivalent to the negation of the opposite:

$$\begin{aligned}
(\forall x)\,Fx &\equiv \sim(\exists x)\sim Fx\\
(\forall x)\sim Fx &\equiv \sim(\exists x)\,Fx\\
(\exists x)\,Fx &\equiv \sim(\forall x)\sim Fx\\
(\exists x)\sim Fx &\equiv \sim(\forall x)\,Fx
\end{aligned}$$

The other relationships can be shown to hold. But, at the moment we will defer the proof to the section on formal modal relations, §12.3.

We also set out another form of the Square which is derived from Aristotle's system of syllogistic.

2.

$$\begin{tikzcd}
(\forall x)(Fx \supset Gx) \arrow[drr, leftrightarrow] & & (\forall x)(Fx \supset \sim Gx) \arrow[dll, leftrightarrow, "MQT\text{-}Contradictories" description] \\
(\exists x)(Fx \,\&\, Gx) & & (\exists x)(Fx \,\&\, \sim Gx)
\end{tikzcd}$$

Once again we can use equivalences and replacement to show how each formula is equivalent to the negation of its diagonal opposite.

A:	All F s are G s	
	$(\forall x)(Fx \supset Gx)$	
	$(\forall x) \sim(Fx \,\&\, \sim Gx)$	$p \supset q \;.\!\equiv\!.\; \sim(p \,\&\, \sim q)$
	$\sim(\exists x)(Fx \,\&\, \sim Gx)$	QN

Here the first translation is the one we are familiar with. The second comes from the first by using the propositional equivalence shown, where we put "*Fx*" for "*p*" and "*Gx*" for "*q*". The third comes from the second by QN. If you work through and understand the **E**, **I**, **O** cases below you will have the idea.

E:	No F s are G s	
	$(\forall x)(Fx \supset \sim Gx)$	
	$(\forall x) \sim(Fx \,\&\, Gx)$	$p \supset \sim q \;.\!\equiv\!.\; \sim(p \,\&\, q)$
	$\sim(\exists x)(Fx \,\&\, Gx)$	QN
I:	Some F s are G s	
	$(\exists x)(Fx \,\&\, Gx)$	
	$(\exists x) \sim(Fx \supset \sim Gx)$	$p \,\&\, q \;.\!\equiv\!.\; \sim(p \supset \sim q)$
	$\sim(\forall x)(Fx \supset \sim Gx)$	QN
O:	Some F s are not G s	
	$(\exists x)(Fx \,\&\, \sim Gx)$	
	$(\exists x) \sim(Fx \supset Gx)$	$p \,\&\, \sim q \;.\!\equiv\!.\; \sim(p \supset q)$
	$\sim(\forall x)(Fx \supset Gx)$	QN

Note that only one of the formal modal relations from the first square of opposition survives in the second square. MQT-Contrariety, MQT-Implication, MQT-Sub-contrariety have all vanished.

It has often been argued that in ordinary language the **A E I** and **O** square of opposition has all the relations set out in **1** above. These are lost on translation into MQL. Indeed, if we agree that:

Every F is G → $(\forall x)(Fx \supset Gx)$
No F is G → $(\forall x)(Fx \supset \sim Gx)$

then it is clear that the failure of the MQL **A** and **E** formulae to be MQT-Contrary does not show that the English **A** and **E** are not contrary. Also the failure of the MQL **A** and **E** to MQT-Imply the MQL **I** and **O**, respectively, does not show that the ordinary language implications fail.

Why is it so? Work it out for yourself by looking back to Chapter 7.

Sub-Contrariety is more difficult, because most would agree that:

Some F is G ↔ $(\exists x)(Fx \,\&\, Gx)$
Some F is not G ↔ $(\exists x)(Fx \,\&\, \sim Gx)$

In this case, failure of sub-contrariety in MQT should show failure in English.

Aristotle, in his logic (See Appendix 1), assumed the contrariety of **A** and **E** and the necessary implication of **I** by **A**, and of **O** by **E**. He seems also to have assumed the sub-contrariety of **I** and **O**. Aristotle's assumptions have been accounted for by what is called the *Existential Viewpoint*. If such assumptions are abandoned in favour of what holds in MQT, in square **2**, then we have what is called the *Hypothetical Viewpoint*.

NOTES

The three vital facts, *A* to *C*, follow from theorems in §4.6 of Alonzo Church's *Introduction to Mathematical Logic* Vol. 1. We have set these facts out in such a way that students will look not only at the upper limit expansion of a form, because it may be possible to find a counterexample in an expansion for a smaller world. If the expansion at the upper limit is a tautology, then, of course, the formula is an MQT-Necessity.

EXERCISE 11.3

1. Test each of the following wffs of MQL for MQT-Necessity.
   - (a) $(\exists x)(Fx \vee \sim Fx)$
   - (b) $(\forall x)Fx \supset (\exists x)Fx$
   - (c) $\sim(\exists x) \sim Fx \equiv (\forall x)Fx$
   - (d) $(\exists x)(Fx \;\&\; Fa) \equiv ((\exists x)Fx \;\&\; Fa)$
   - (e) $(\forall x)(Fx \vee Fa) \equiv ((\forall x)Fx \vee Fa)$

2. Show, by eliminating quantifiers, which of the following is a counterexample to the formula beside it.

(a)

	F
a	1
b	1

$(\forall x)Fx \equiv (\exists x)Fx$

(b)

	F	*G*
a	1	1
b	0	1

$(\forall x)(Fx \supset Gx) \supset (\forall x)(Gx \supset Fx)$

(c)

	F	*G*
a	1	0
b	0	1

$(\exists x)(Fx \;\&\; Gx) \supset (\exists x)(Gx \;\&\; Fx)$

(d)

	F	*G*
a	0	1
b	0	1

$(\forall x)(Fx \supset Gx) \supset \sim(\forall x)(Fx \supset \sim Gx)$

(e)

	F	*G*
a	1	0
b	1	1

$((\exists x)Fx \;\&\; (\exists x)Gx) \supset (\exists x)(Fx \;\&\; Gx)$

3. Let:

a = Anne
b = Bruce
c = Chris
Cx = x cheered
Lx = x cheered loudly
Rx = x ran
Sx = x ran swiftly
Dx = x is a dog
Ax = x is an animal

Given the dictionary above, which of the following tables describe possible worlds, and which do not? You will have to use your logical intuitions.

(a)

	C	*L*
a	1	1
b	0	0

(b)

	C	*L*	*R*	*S*
a	1	1	0	0
b	0	0	1	1

(c)

	C	*L*	*R*	*S*
a	0	1	1	1
b	0	0	1	0

(d)

	R	*S*	*D*	*A*
b	1	1	0	0
c	1	0	1	1

(e)

	D	*A*
c	1	0

(f)

	R	*S*	*C*	*L*
c	1	0	1	0

*4. Write out the version of fact *B* which would apply to *MQT-Contradictions.*

11.4 VALIDITY IN MQT

The definitions of validity given in §4.3 are completely general. Our main interest in this section is with the validity and invalidity of argument forms, in particular *MQL-argument-forms.* In any finite world every MQL-argument-form has an equivalent argument-form which is found by expanding the premises and the conclusion. We will call this the expansion of the argument-form. In MQT we need take account of finite worlds only, because of fact *A*.

Definition: An MQL-argument-form is valid iff all its expansions are valid.

Expansions of MQL-argument-forms can be tested for validity in PC. Consider argument-form (1).

$$\begin{array}{l}\sim(\forall x)\,Fx\\ \hline \therefore\ (\exists x)\sim Fx\end{array}\qquad\qquad\begin{array}{l}\text{Not every item is } F\\ \hline \therefore\ \text{At least one item is non-}F.\end{array}\tag{1}$$

In every one-item world, where we name that item "*a*", the argument-form expands to (2).

$$\begin{array}{l}\sim Fa\\ \hline \therefore \sim Fa\end{array}\tag{2}$$

This expanded argument-form can be evaluated by PC. It is valid. In every two-item world, where we name the items "*a*" and "*b*", argument-form (1) expands to (3).

$$\begin{array}{l}\sim(Fa\ \&\ Fb)\\ \hline \therefore \sim Fa \vee \sim Fb\end{array}\tag{3}$$

(3) is valid in PC.

The second of the facts about MQT, fact *B*, set out in §11.3, transfers across to argument-forms to give a finite upper limit on testing for validity.

If an MQL-argument-form contains *n* predicate letters, then the argument-form is valid iff every expansion of the argument-form up to 2^n items is PC-valid.

Since argument (1) has one predicate letter and is PC-valid for one and two item expansions, it follows that (1) is MQT-Valid.

Example (4) is invalid because its expansion for two items is invalid.

$$\begin{array}{l}(\exists x)Fx\\ \hline \therefore\ (\forall x)Fx\end{array}\qquad\qquad\begin{array}{l}Fa \vee Fb\\ \hline \therefore\ Fa\ \&\ Fb\end{array}\tag{4}$$

So we can generate the following counterexample:

	F
a	1
b	0

Fact *C*, as set out in §11.3, provides a further upper limit on expansions for argument testing. To apply fact *C* here we stipulate that the argument-form meets the *Short Cut Condition* (**SCC**).

SCC: *Each premise and the conclusion must have no more than one quantifier, and if there is a quantification it must be the main operator.*

If the argument-form passes the **SCC**, we then count the number of existentially quantified premises, and add 1 if the conclusion is universally quantified. The resulting number, or 1 if the number is zero, gives the upper limit on the expansions needed to show validity.

Method:

1. Check that the argument meets SCC.
2. Count the existentially quantified premises; if the conclusion is universally quantified, add 1.
3. (a) If the total from step 2 is one or zero, check the argument in its one item expansion.
   (b) If the total from step 2 is greater than one, check the expansions from one up to that number.

So to test the following argument-form we need look only at its one-item expansion, set out beside it:

$$\begin{array}{c} (\forall x)(Fx \supset Gx) \\ (\forall x)(Gx \supset Hx) \\ \hline \therefore (\forall x)(Fx \supset Hx) \end{array} \qquad\qquad \begin{array}{c} Fa \supset Ga \\ Ga \supset Ha \\ \hline \therefore\ Fa \supset Ha \end{array}$$

The argument-form is MQT-Valid, because our count gives 1, and the one-item expansion is PC-valid.

Apart from this short cut, and in some cases with it, the search through expansions can be very inefficient. More generally efficient methods are available, especially in the form of truth-trees. Diagrams can also be used. We will look at both in later chapters.

Finally, we must take note of the difference between the invalidity of an argument-form and the invalidity of an argument.

When we test an argument, not an argument-form, we cannot be sure we have a counterexample until a countermodel is seen to be a *possible* world. Consider the following example:

> All bachelors are male. All unmarried men are male. So, all bachelors are unmarried.

This symbolizes as:

$$\begin{array}{rl} & (\forall x)(Bx \supset Mx) \\ & (\forall x)(\sim Wx \supset Mx) \\ \hline \therefore & (\forall x)(Bx \supset \sim Wx) \end{array}$$

Bx = x is a bachelor
Mx = x is male
Wx = x is married

To test this we need only test:

$$\begin{array}{rl} & Ba \supset Ma \\ & \sim Wa \supset Ma \\ \hline \therefore & Ba \supset \sim Wa \end{array}$$

In MQT we can set out the following *purported* counterexample, or countermodel.

	B	M	W
a	1	1	1

Since this does not describe a possible world, it does not demonstrate the invalidity of the argument. It is a countermodel but not a counterexample.

EXERCISE 11.4

1. Test the following argument-forms for validity. If any is invalid set out a counterexample.

(a)
$$\begin{array}{rl} & (\forall x)(Fx \supset Gx) \\ & (\forall x)(Hx \supset Gx) \\ \hline \therefore & (\forall x)(Fx \supset Hx) \end{array}$$

(b)
$$\begin{array}{rl} & (\forall x)(Fx \supset Gx) \\ & (\exists x)(Gx \,\&\, Hx) \\ \hline \therefore & (\exists x)(Fx \,\&\, Hx) \end{array}$$

(c)
$$\begin{array}{rl} & (\exists x)\, Fx \\ & (\exists x)\, Gx \\ \hline \therefore & (\exists x)(Fx \,\&\, Gx) \end{array}$$

(d)
$$\begin{array}{rl} & (\forall x)\, Fx \supset (\forall x)\, Gx \\ \hline \therefore & (\forall x)(Fx \supset Gx) \end{array}$$

(e)
$$\begin{array}{rl} & (\forall x)\, Fx \,\&\, (\forall x)\, Gx \\ \hline \therefore & (\forall x)(Fx \,\&\, Gx) \end{array}$$

(f)
$$\begin{array}{rl} & (\forall x)(Fx \supset Gx) \\ & (\forall x)(Hx \supset \sim Gx) \\ \hline \therefore & (\forall x)(Fx \supset \sim Hx) \end{array}$$

(g)
$$\begin{array}{rl} & (\exists x)(Fx \,\&\, Gx) \\ & (\forall x)(Gx \supset Hx) \\ \hline \therefore & (\exists x)(Fx \,\&\, Hx) \end{array}$$

(h)
$$\begin{array}{rl} & (\forall x)(Fx \supset \sim Gx) \\ & (\exists x)(Fx \,\&\, Hx) \\ \hline \therefore & (\exists x)(Hx \,\&\, \sim Gx) \end{array}$$

(i)
$$\begin{array}{rl} & (\forall x)(Fx \supset Gx) \\ & (\forall x)(Gx \supset Hx) \\ \hline \therefore & (\exists x)(Hx \,\&\, Fx) \end{array}$$

(j)
$$\begin{array}{rl} & (\forall x)(Fx \supset Gx) \\ & (\forall x)(Gx \supset \sim Hx) \\ \hline \therefore & (\forall x)(Hx \supset \sim Fx) \end{array}$$

2. Translate the following arguments into MQL using only the dictionary provided. Test each for validity, and where necessary distinguish between MQT-Validity and validity.

(a) Some sound arguments have been used in public debate. Sound arguments are all worthwhile. So, some arguments used in public debate are worthwhile.
(Sx = x is a sound; Ux = x has been used in public debate; Wx = x is worthwhile; Ax = x is an argument.)

(b) Some policies tend to centralise authority. All policies require careful planning. So, some things which require careful planning tend to centralise authority. (Px = x is a policy; Cx = x tends to centralise authority; Rx = x requires careful planning.)

(c) Since all the crates from the terminal are stored in the shed, and some hexagonal crates are stored in the shed, it follows that some of the crates from the terminal are hexagonal.
($Cx = x$ is a crate from the terminal; $Sx = x$ is stored in the shed; $Hx = x$ is a hexagonal crate.)

(d) Some average students will be almost certain to pass. Why? Because any student who works very hard will be almost certain to pass, and some average students work very hard.
($Ax = x$ is an average student; $Px = x$ will be almost certain to pass; $Hx = x$ works very hard; $Sx = x$ is a student.)

* (e) Every person whose vehicle is registered in either April or May, and only such persons, will be issued with petrol rationing coupons on Wednesdays. Jane has a vehicle which is registered in April, but Susan's vehicle is registered in July. So, Jane will be issued with petrol rationing coupons on Wednesdays but Susan will not.
(a = Jane; b = Susan; $Ax = x$ has a vehicle registered in April; $Mx = x$ has a vehicle registered in May; $Jx = x$ has a vehicle registered in July; $Rx = x$ will be issued with petrol rationing coupons on Wednesdays; $Px = x$ is a person.)

11.5 RESTRICTED UNIVERSES OF DISCOURSE

It is sometimes useful, when analysing arguments or propositions, to assume that everything in all worlds has a common property. We might assume that every item is a person. In this way we *restrict the universe of discourse* to persons. Such a restriction should be made clear by an entry in the dictionary:

Universe = persons.

The quantifiers are then altered. The universal quantifier means:

Each and every person
or *Everyone*
or *Everybody*

The existential quantifier means:

At least one person
or *Someone*
or *Somebody*

In general, if we have:

Universe = *F*s

then:

$(\forall x)$ means *every F*
$(\exists x)$ means *at least one F*

Before restricting the universe of discourse make sure that all quantifiers in an argument will be able to bear the restriction. For example, there is no worry with:

All propositions are either true or false.
No proposition is both true and false.
So, every proposition is either true or false but not both.

Notice how every quantity is a quantity of propositions. So we could set up the dictionary and symbolize the argument as follows:

Universe = propositions
Tx = x is true
Fx = x is false

$$\begin{array}{rl} & (\forall x)(Tx \lor Fx) \\ & \sim (\exists x)(Tx \,\&\, Fx) \\ \hline \therefore & (\forall x)(Tx \not\equiv Fx) \end{array}$$

on the other hand, consider the argument:

All cats are mammals, but no crabs are mammals. So, no cats are crabs.

We cannot restrict the universe of discourse here to cats, or to crabs.

Restricting a universe of discourse to some property, P, is not without logical effect. It is not simply a matter of convenience.

Remember, in § 10.4, we said that our worlds in MQT were *non-empty*. Each domain contains at least one item. So if we restrict our universe of discourse to property P we are automatically assuming that there is at least one P in every world. If we do not restrict the universe it is an open question as to whether or not there are any items with the property P in any world.

Restricting the universe of discourse can have a definite effect on the validity of arguments. Consider the argument:

Since everyone is mortal, someone is mortal.

We set out two dictionaries and the two symbolizations

Universe = persons
$Mx = x$ is mortal

$$\begin{array}{rl} & (\forall x)\, Mx \\ \hline \therefore & (\exists x)\, Mx \end{array} \qquad (1)$$

$Px = x$ is a person
$Mx = x$ is mortal

$$\begin{array}{rl} & (\forall x)(Px \supset Mx) \\ \hline \therefore & (\exists x)(Px \,\&\, Mx) \end{array} \qquad (2)$$

Now argument (1) is valid, simply because every world has at least one person in it, and the premise says that all are mortal. But argument (2) is invalid. A counterexample is to be found in the following world where the argument expansion is as set out:

	P	M
a	0	1

$$\begin{array}{rl} & Pa \supset Ma \\ \hline \therefore & Pa \,\&\, Ma \end{array} \qquad (3)$$

(3) is invalid. It's invalid simply because there are no persons in the counterexample. If we added as a premise, the proposition $(\exists x)\, Px$ then the resulting argument would be valid.

One way of summarizing this is to say that although our worlds are non-empty, our properties can be empty (unless there is a restricted universe of discourse).

NOTES

One way of giving individual constants existential import, and of making $(\exists x)$ a truly *existential* quantifier, would be to restrict the universe of discourse to existing items.

EXERCISE 11.5

1. Translate the following into MQL using only the dictionary provided.

Universe = Persons
Px = x is perfect
Kx = x is kind
Tx = x is thoughtful
Sx = x is sensible
Mx = x makes mistakes
Ax = x is altruistic

(a) Nobody is perfect
(b) Someone is kind.
(c) Everyone who is thoughtful is kind.
(d) Some people are thoughtful and some are not.
(e) If someone is perfect then he or she is kind.
(f) Everyone makes mistakes.
(g) Some people make mistakes even though they are sensible and kind.
(h) Everyone who is unkind makes mistakes.
(i) Altruistic people are always kind.
(j) People who are not thoughtful are not sensible.

2. Translate all the propositions in 11.3 (1) into MQL but change the first entry in the dictionary to : *Universe* = Propositions.

3. Translate and test argument 11.4 (2(e)) but let *Universe* = Persons.

11.6 MORE COMPLEX CASES IN MQT

Propositional variables can occur in formulae of MQT. (1), (2) and (3) are formulae of MQT.

$$(\exists x)\, Fx \supset p \tag{1}$$

$$(\exists x)(Fx \supset p) \tag{2}$$

$$p \supset (\forall x)(Fx \supset q) \tag{3}$$

To work out the truth-values of these in any world we need values for the propositional variables. For example:

(A)

	F	p	q
a	1	0	1
b	0		

The expansion of (1) for world (A) is (4):

$$(Fa \vee Fb) \supset p \tag{4}$$

We simply eliminate the existential quantifier. (4) is false

$$(1 \vee 0) \supset 0 \;=\; 0$$

The expansion of (2) is (5). The elimination of the quantifier is more tricky here. Compare it with (4). Look at the *differing scopes* of $(\exists x)$

$$(Fa \supset p) \vee (Fb \supset p) \tag{5}$$

(5) is true:

$$(1 \supset 0) \vee (0 \supset 0) \;=\; 0 \vee 1 \;=\; 1$$

The expansion of (3) is (6), and (6) is true.

$$p \supset ((Fa \supset q) \,\&\, (Fb \supset q)) \tag{6}$$

The same applies for propositional constants.

Let R = It's raining.
Fx = x is a frog

We would read (7) as (8)

$(\exists x)\, Fx \supset R$ (7)
If there is a frog then it's raining (8)

To test (9) for MQT-Necessity we check the expansions of (9) for one and two item worlds. Both expansions, (10) and (11) are tautologies.

$$(\exists x)(Fx \supset R) \quad \supset \quad ((\forall x)\, Fx \supset R) \tag{9}$$

$$(Fa \supset R) \quad \supset \quad (Fa \supset R) \tag{10}$$

$$((Fa \supset R) \vee (Fb \supset R)) \quad \supset \quad ((Fa \,\&\, Fb) \supset R) \tag{11}$$

So (9) is an MQT-Necessity.

The formation rules for MQL allow for formulae like (12) and (13).

$$(\exists x)(Fx \,\&\, (\forall y)\, Gy) \tag{12}$$
$$(\exists x)(\exists y)(Fy \,\&\, Gx) \tag{13}$$

Our main concern is how to eliminate quantifiers inside the scope of another quantifier. We begin by eliminating those quantifiers which have no quantifier in their scope. We do this repeatedly until all quantifiers have been eliminated. For example, we expand (12) for *a, b*, step by step. We eliminate $(\forall y)$ to get (14)

$$(\exists x)(Fx \,\&\, (Ga \,\&\, Gb)) \tag{14}$$

Then we eliminate $(\exists x)$ to get (15).

$$(Fa \,\&\, (Ga \,\&\, Gb)) \vee (Fb \,\&\, (Ga \,\&\, Gb)) \tag{15}$$

We begin with the *innermost* quantifiers. With (13) we begin with $(\exists y)$ to get (16).

$$(\exists x)((Fa \,\&\, Gx) \vee (Fb \,\&\, Gx)) \tag{16}$$

Take care. Only y was replaced in the scope. Then we get (17).

$$((Fa \,\&\, Ga) \vee (Fb \,\&\, Ga)) \vee ((Fa \,\&\, Gb) \vee (Fb \,\&\, Gb)) \tag{17}$$

Compare (13), (16) and (17) again. When we itemize the scope of a quantification *only the variables bound by the quantifier are replaced.*

We have seen that some formulae of MQL contain *vacuous quantifiers*. When such formulae are expanded for any universe the vacuous quantifiers are simply dropped. For example, (19) is the expansion of (18) for *a, b*.

$$(\forall x)(\exists y)Fx \tag{18}$$

$$Fa \,\&\, Fb \tag{19}$$

We have also seen that some formulae are *open*. In order to deal with open formulae in MQT they must first be *universally closed.* So to expand (20) for *a, b* we first close it to get (21), and then expand to get (22).

$$Fx \supset Gx \tag{20}$$

$$(\forall x)(Fx \supset Gx) \tag{21}$$

$$(Fa \supset Ga) \,\&\, (Fb \supset Gb) \tag{22}$$

The principle behind treating free occurrences of variables as though they were universally quantified is that this is what happens in arithmetic. We treat (23) as though it were (24)

$$x + x = 2x \tag{23}$$

$$(\forall x)(x + x = 2x) \tag{24}$$

when the universe of discourse is restricted to numbers.

EXERCISE 11.6

1. Expand the following formulae for *a*, *b*.
   - (a) $P \supset (\exists x)(Fx \ \& \ Gx)$
   - (b) $(\exists x)(P \supset (Gx \ \& \ Fx))$
   - (c) $(\forall x)(Gx \supset Fx) \supset Q$
   - (d) $(\forall x)((Gx \supset Fx) \supset Q)$
   - (e) $Ga \supset Gx$
   - (f) $(\forall x)(\exists y)(P \supset (Gx \ \& \ Fy))$
   - (g) $(\exists y)(P \supset (Ga \vee Gx)) \supset Fx$
   - (h) $((\exists x)\, Fx \supset p) \equiv (\forall x)(Fx \supset p)$
   - (i) $((\forall x)\, Fx \supset p) \equiv (\exists x)(Fx \supset p)$
   - (j) $(p \supset (\forall x)\, Fx) \equiv (\forall x)(p \supset Fx)$

2. Translate the following into QL using only the dictionary provided.

   T = the train comes
   l = the line
   Sx = x is sick
   Dx = x is destroyed
   Gx = x will go
   Px = x is a person

   - (a) If the train comes then the sick will all go.
   - (b) If the train does not come then some of the sick will not go.
   - (c) If the train comes then the line is not destroyed.
   - (d) Some people are not sick and they will go.
   - (e) If everything which will go is destroyed then the train will not come.

3. Test the following forms for MQT-Necessity. If there is a counterexample, set it out.
   - (a) $(\forall x)\, Fx \supset Fy$
   - (b) $Fx \supset (\forall x)\, Fx$
   - (c) $((\forall x)\, Fx \supset p) \equiv (\exists x)(Fx \supset p)$
   - (d) $Gx \supset Gx$
   - (e) $Gx \supset (\exists x)\, Gx$
   - (f) $(\exists x)\, Gx \supset Gx$

4. Show, by quantifier elimination, which of the following are counterexamples to the associated formulae.

(a)

p	q
1	1

	F
a	0

$(p \ \& \ q) \supset (\exists x)\, Fx$

(b)

p	q
0	0

	F
a	0
b	1

$(\exists x)\, Fx \supset (p \vee q)$

(c)

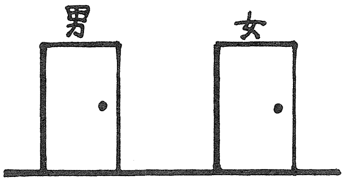
p
1

	F	G
a	0	1
b	1	0

$(\forall x)(Fx \supset Gx) \supset \sim p$

5. Provide a counterexample for each of the following formulae, verifying your counterexample in each case.
   - (a) $(\exists x)\, Fx \supset (\exists x)\, Gx$
   - (b) $(\forall x)(Fx \vee Gx)\ .\!\supset\!.\ (\forall x)\, Fx \vee (\forall x)\, Gx$
   - (c) $(\forall x)(\, Fx \vee Gx) \equiv (\forall x)\, Fx$
   - (d) $(\forall x)\, Fx \supset (\exists x) \sim\! Fx$
   - (e) $(\exists x)(Fx \supset Gx) \supset (\exists x)(Fx \;\&\; Gx)$

Puzzle 11 On arriving at a hotel in Japan a tourist heads straight for the rest rooms. He finds his way there but is unable to decide which of two doors to choose because the "Men" and "Women" signs are written in Japanese. Fortunately, a Japanese gentleman is standing nearby, and the visitor recognizes him as being one of two identical twins. Although he doesn't know which twin it is, he does know that one of the twins always tells the truth and the other twin always lies.



Given that the twin will answer one question only, and that with only a "Yes" or "No", what question would you ask if you were the tourist?

11.7 SUMMARY

English sentences may often be reworded in terms of the **A**, **E**, **I**, **O** forms. Such forms are standardly translated into MQL as follows:

A	Every *F* is *G*	$(\forall x)\ (Fx \supset Gx)$
E	No *F* is *G*	$(\forall x)\ (Fx \supset \sim\! Gx)$ or $\sim\!(\exists x)\ (Fx \;\&\; Gx)$
I	Some *F* is *G*	$(\exists x)\ (Fx \;\&\; Gx)$
O	Some *F* is not *G*	$(\exists x)\ (Fx \;\&\; \sim\! Gx)$

In English the *existential viewpoint* is often adopted for these propositions e.g., in the above cases it is assumed that an item with the property *F* exists: from this viewpoint various other relations may be added to the contradictoriness relation on the square of opposition (see § 10.6) e.g., **A** and **E** are contraries, **A** implies **I**, **E** implies **O**. In MQT however, the *hypothetical viewpoint* is taken which does not adopt such existential presuppositions: while the contradictoriness relation holds for the MQT square of opposition, the other relations do not.

Any occurrence of an IV either in, or within the scope of, a quantification with respect to it, is *bound*. An occurrence which is not bound is *free*. Binding lines may be used to join variables bound by the same quantifier, and free variables may be underlined, e.g., $(\forall x)\, Fx \supset Gx$ $\quad (\forall x)\ (\exists y)\ (Fx \supset Gy)$

IVs can be bound by only one quantifier (the first that binds it in an assembly line). A quantifier that binds nothing is *vacuous* e.g., $\forall$ in $(\forall x)\,(\exists x)\, Fx$. A formula is *open* iff it has at least one free IV. Prefixing an open formula with universal/existential quantifications to bind all the free IVs gives universal/existential *closure* e.g., Fx & Gx is universally closed to $(\forall x)$ $(Fx$ & $Gx)$.

MQL-formulae are either *forms* (these have no dictionary) or *sentences* (these have a full dictionary, and express propositions). A *form* is an *MQT-Necessity, MQT-Contradiction* or *MQT-Contingency* according as it is true in all, none or just some worlds. A *proposition* is an *MQT-Necessity, MQT-Contradiction* or *MQT-Indeterminacy* according as its explicit MQL-form is an MQT-Necessity, MQT-Contradiction or MQT-Contingency.

If α is an MQL-form with n different predicate letters then α is an MQT-Necessity/MQT-Contradiction iff every expansion of α up to 2^n items is a tautology/PC-contradiction. Consider any form $(Q\nu)$ ... $(Q\omega)\alpha$ where the Qs are non-vacuous quantifiers, of which n are universal/existential, and α, the scope of $(Q\omega)$, is quantifier-free: then the form is an MQT-Necessity/MQT-Contradiction iff every expansion up to n items (or one item if $n = 0$) is a tautology/PC-contradiction.

The following equivalences are examples of *Quantifier Negation (QN):*

$$\sim(\exists x)\, Fx \Leftrightarrow (\forall x) \sim Fx$$
$$\sim(\forall x)\, Fx \Leftrightarrow (\exists x) \sim Fx$$

An MQL- argument-form is *valid* iff there is no world with its premises true and conclusion false; otherwise it is *invalid.* If an MQL-argument-form contains n different predicate letters then it is valid iff every expansion up to 2^n items is PC-valid. *If* an MQL-argument-form meets the *Short Cut Condition* (*SCC*: each premise and the conclusion has at most one quantifier, and if there is a quantification it must be the main operator) *then* count the $\exists$s in the premises and add 1 if there is a $\forall$ in the conclusion to give a total of t: if $t = 0$ or 1 then only the one-item expansion need be tested; otherwise test up to just the t-item expansion.

A *countermodel* to an *argument* is a world where the premises are true and the conclusion is false: for this to be a *counterexample* the world must be a *possible* world. As for PC, an argument is valid iff it has no counterexample.

If in translation we *restrict the universe of discourse* to a certain type of item (e.g., people), this should be noted at the top of the dictionary. Although worlds are non-empty, properties may be empty.

In expanding complex formulae the following bottom-up procedure is recommended: *at each step eliminate just those quantifiers with no quantifiers in their scope.* Note that when itemizing the scope of a quantification, *only the variables bound by the quantifier are replaced.* Vacuous quantifiers may simply be dropped. Open formulae must be universally closed before expanding.

12 MQT Trees

12.1 TRUTH TREES FOR MQT

The truth-tree method of testing formulae and arguments can easily be extended from PC to MQT. Truth-trees provide us with a way of *searching for a counterexample.* If there is no counterexample, the tree closes.

In order to extend the truth-trees to MQT we need four new rules, two for each new operator. We need rules for universal quantification and its negation, and rules for existential quantification and its negation. Otherwise, everything is the same as for PC. We retain all the PC rules, the same general rule for the closure of a path, and the same general approaches to testing logical necessities, contradictions, modal relations and arguments.

We begin by setting out the rules for the negations of the quantifications. These rules are simple and similar. Then we turn to the rules for universal and existential quantifications.

We saw in §11.3 that (1) and (2) are necessary truths of MQT

$$\sim(\exists x)\, Fx \equiv (\forall x) \sim Fx \tag{1}$$
$$\sim(\forall x)\, Fx \equiv (\exists x) \sim Fx \tag{2}$$

Our rules are similar. Both are called *Quantifier Negation*, and both are used to eliminate a " $\sim$ " from the front of a quantified formula.

Quantifier Negation (QN):

$\surd\ \sim(\exists \nu)\, \alpha$	$\surd\ \sim(\forall \nu)\, \alpha$
$(\forall \nu) \sim \alpha$	$(\exists \nu) \sim \alpha$

Examples:

$\surd$	1. $\sim(\exists x)\, Fx$		$\surd$	1. $\sim(\forall x)(Fx \supset Gx)$	
	2. $(\forall x) \sim Fx$	1, QN		2. $(\exists x) \sim (Fx \supset Gx)$	1, QN

In these rules it should be noted that α is the scope of the quantifier. Sometimes we say "Slide the tilde through the quantification and it changes the quantifier".

The rule for the Universal Quantification is called *Universal Instantiation.* For example:

Let: $Fx =$ *x is a frog.*

Clearly we read (3) as (4)

$(\forall x)\ Fx$ (3)

Everything is a frog (4)

Now, if everything is a frog, then *a* is a frog, *b* is a frog, etc. Every item in the domain of quantification is a frog:

Fa
Fb
Fc
$\vdots$

So, all of these follow from (3). Each is an itemization of the scope of $(\forall x)$. Since every itemization of the scope of $(\forall x)$ follows from (3) we may apply *Universal Instantiation* to (3) in a tree *any number of times.* So, in applying this rule we do *not* tick the formula in the usual way (because we might want to use it again).

The general form of Universal Instantiation (UI) is as follows:

Universal Instantiation (UI)

$(\forall \nu)\ \phi\nu$
$\vdots$
$\phi\kappa$ where $\phi\kappa$ is any itemization of $\phi\nu$

When $\phi\kappa$ is the itemization of $\phi\nu$ to κ by UI we say that $(\forall\nu)\ \phi\nu$ has been *Universally Instantiated, with respect to* κ, *to* $\phi\kappa$. We use a special marking for UI. For example:

$a\backslash$	1.	$(\forall x)\ Fx$	
	2.	Fa	1, UI

We use a backslash and specify the individual constant with respect to which the formula is Universally Instantiated. Here is another example:

$cb\backslash$	1.	$(\forall x)(Fx \vee Gx)$	
	2.	$Fb \vee Gb$	1, UI
	3.	$Fc \vee Gc$	1, UI

So, each time UI is applied, we add an individual constant to the left of the list. The following are all cases of UI.

$a\backslash$	1.	$(\forall x) \sim Gx$	
	2.	$\sim Ga$	1, UI

$b\backslash$	1.	$(\forall x)(Fx \supset (\exists y)\ Gy\ )$	
	2.	$Fb \supset (\exists y)\ Gy$	1, UI

$cba\backslash$	1.	$(\forall y)\ Gy$	
	2.	Ga	1, UI
	3.	Gb	1, UI
	4.	Gc	1, UI

The rule for the Existential Quantification is called *Existential Instantiation* (EI). This is slightly more tricky than UI. We read (5) as (6)

$(\exists x)\ Fx$ (5)

At least one thing is a frog (6)

Provided "*a*" does not already occur anywhere in this path we may call this item "*a*". In other words, we may *name* the item "*a*". So from (5) it follows that (7), provided *a* is new to the path.

Fa (7)

The general rule is as follows

Existential Instantiation (EI):

$\surd\ (\exists \nu)\ \phi\nu$

$\phi\kappa$ where $\phi\kappa$ is an itemization of $\phi\nu$ to an individual constant, κ, *new* to this path of the tree.

When $\phi\kappa$ is the itemization of $\phi\nu$ to κ by EI we say that $(\exists\nu)\phi\nu$ has been *existentially instantiated, with respect to* κ, *to* $\phi\kappa$.

Each application of EI requires a *fresh* individual constant, since we must not assume that an already mentioned item satisfies the existential expression. For example:

$\surd$	1.	$\sim((\exists x)\ Fx \supset Fa)$	NF
$b\ \surd$	2.	$(\exists x)\ Fx$	1, PC
	3.	$\sim Fa$	
	4.	Fb	2, EI

First note that the ticking off of 2 has the individual constant b beside the tick. Secondly, we cannot instantiate 2. with a because that would mean that we had *assumed* that the F was a, but we must leave the question open. We use "b". Note that in using "b" we do not assume that "b" and "a" must refer to different items: it is possible for one item to have several names.

Remember that PC rules may be used only when a PC operator is the main operator. The same principle operates in MQT. **The rules UI and EI can be used only when the quantification being resolved is the main operator.** You must not instantiate through a tilde.

For example, the Rules UI and EI may be applied to (8), (9) and (10), but must not be applied to (11), (12) or (13).

$(\exists x) \sim Fx$ (8)

$(\forall x)(Fx \supset p)$ (9)

$(\exists x)((Fx \ \&\ Gx) \supset Fa)$ (10)

$\sim(\exists x)\ Fx$ (11)

$(\exists x)\ Fx \supset p$ (12)

$Fa \supset (\forall x)\ Fx$ (13)

In (11) the main operator is $\sim$, and QN must be used. In (12) and (13) the main operator is $\supset$, and the PC rule for $\supset$ must be used.

We now rewrite both UI and EI with the marking and ticking displayed.

UI $\kappa \ldots \backslash$ $(\forall\nu)\ \phi\nu$

$\phi\kappa$ where $\phi\kappa$ is an itemization of $\phi\nu$ to *any* individual constant κ.

EI $\kappa\ \surd$ $(\exists\nu)\ \phi\nu$

$\phi\kappa$ where $\phi\kappa$ is an itemization of $\phi\nu$ to an individual constant, κ, *new* to the path of the tree.

Look carefully at the following applications of the rules to relatively complex formulae. Remember that in an itemization of the scope of a quantification with respect to ν only free occurrences of ν in the scope are replaced.

$b \; \surd$ 1. $(\exists x)(Fx \;\&\; (\exists y)\, Gy)$
2. $Fb \;\&\; (\exists y)Gy$ 1, EI

$b \; \backslash$ 1. $(\forall y)((\forall x)Fx \supset Fy)$
2. $(\forall x)Fx \supset Fb$ 1, UI

$a \; \backslash$ 1. $(\forall x)(Fx \supset (\exists x)Fx)$
2. $Fa \supset (\exists x)Fx$ 1, UI

$a \; \surd$ 1. $(\exists x)((\exists y)(Gy \;\&\; Fy) \vee Hx)$
2. $(\exists y)(Gy \;\&\; Fy) \vee Ha$ 1, EI

EXERCISE 12.1

1. Does the second of each of the following pairs of formulae follow from the first by the rule for trees as annotated? If not, why not?

(a) $(\forall x)(Sx \supset Fx)$
$(Sa \supset Fa)$ UI

(b) $(\exists x) \sim Mx$
$\sim Ma$ EI

(c) $(\forall x)(Sx \supset (Fx \;\&\; Gx))$
$(Sa \supset (Fa \;\&\; Ga))$ EI

(d) $\sim((\exists x)\, Fx \;\&\; Gx)$
$(\forall x) \sim (Fx \;\&\; Gx)$ QN

(e) $(\forall x)(Sx \supset (\forall x)\, Fx)$
$(Sa \supset (\forall x)\, Fx)$ UI

(f) $(\exists x)(Sa \;\&\; (Gx \vee Fx))$
$(Sa \;\&\; (Ga \vee Fa))$ EI

(g) $(\forall x)(Sa \supset (Gx \vee Fx))$
$(Sa \supset (Ga \vee Fa))$ UI

(h) $\sim(\forall x)(Fx \supset p)$
$(\exists x) \sim (Fx \supset p)$ QN

(i) $(\exists x)(Fx \supset p)$
$(Fa \supset p)$ EI

(j) $(p \supset (\forall x)Fx)$
$(p \supset Fa)$ UI

(k) $(\forall x)((\forall x)Fx \supset Gx)$
$(Fa \supset Ga)$ UI

(l) $(\exists x) \sim (\forall y)(Fx \supset Gy)$
$\sim(\forall y)(Fa \supset Gy)$ EI

(m) $((\forall x)\, Fx \supset p)$
$(Fa \supset p)$ UI

(n) $((\forall x)Fx \supset (\forall x)Gx)$
$(Fa \supset Ga)$ UI

(o) $(\exists x)(Fx \supset (\forall y)(Ga \supset Ry))$
$(Fb \supset (\forall y)(Ga \supset Ry))$ EI

(p) $(\forall x)\, Fx \supset Gx$
$(Fa \supset Ga)$ UI

2. Apply the appropriate tree rule to each of the following formulae.

(a) $(\forall x)Fx$
(b) $\sim((\exists x)\, Fx \;\&\; Ga)$
(c) $(\exists x)(Fx \;\&\; Gx)$
(d) $(\exists x)(Fx \vee Ga)$
(e) $(\forall x)\, Fx \supset p$
(f) $(\forall x) \sim (p \supset Gx)$
(g) $(\exists x)\, Fx \vee (\exists x)\, Gx$
(h) $(\exists x)(Fx \vee (\forall x)\, Gx)$
(i) $\sim(\forall x)(Fx \supset Gx)$
(j) $(\exists x)((Fx \vee Gx) \;\&\; Fa)$

12.2 TESTING PROPOSITIONS AND ARGUMENTS

MQT-trees may be used to test formulae for MQT-Necessity and MQT-Contradiction. The same general method is used as was used in PC. We set out three simple examples.

Example 1: To see if $(\forall x)(Fx \supset Fx)$ is an MQT-Necessity.

	$\surd$	1.	$\sim(\forall x)(Fx \supset Fx)$	NF
a	$\surd$	2.	$(\exists x) \sim(Fx \supset Fx)$	1, QN
		3.	$\sim(Fa \supset Fa)$	2, EI
		4.	Fa	PC
		5.	$\sim Fa$	
			×	

So, $(\forall x)(Fx \supset Fx)$ is an MQT-Necessity.

Example 2: To see if $(\forall x)(Fx \supset Gx)$ is an MQT-Necessity.

	$\surd$	1.	$\sim(\forall x)(Fx \supset Gx)$	NF
a	$\surd$	2.	$(\exists x) \sim(Fx \supset Gx)$	1, QN
	$\surd$	3.	$\sim(Fa \supset Ga)$	2, EI
		4.	Fa	PC
		5.	$\sim Ga$	
			↑	

Since the tree will never close $(\forall x)(Fx \supset Gx)$ is not an MQT-Necessity. We can read off a counterexample:

	F	G
a	1	0

In this one item world the expansion of $(\forall x)(Fx \supset Gx)$ is false:

$$Fa \supset Ga$$
$$1 \supset 0 \quad = 0$$

Example 3: To see if $(\exists x)(Fx \,\&\, \sim Fx)$ is an MQT-Contradiction

a	$\surd$	1.	$(\exists x)(Fx \,\&\, \sim Fx)$	F
	$\surd$	2.	$Fa \,\&\, \sim Fa$	1, EI
		3.	Fa	2, PC
		4.	$\sim Fa$	
			×	

So, $(\exists x)(Fx \,\&\, \sim Fx)$ is an MQT-Contradiction.

In testing arguments we also follow the same general method as in PC. We set out two simple examples.

Example 4: All students are protesters. Anne is a student. So, Anne is a protester.

$(\forall x)(Sx \supset Px)$

Sa

$\therefore$ Pa

a = Anne
Sx = x is a student
Px = x is a protester

The tree is:

a	\	1.	$(\forall x)(Sx \supset Px)$	P
		2.	Sa	P
		3.	$\sim Pa$	NC
		4.	$Sa \supset Pa$	1, UI
		5.	$\sim Sa \qquad Pa$	4, PC
			× $\qquad$ ×	

Since the tree closes, the argument is valid.

Example 5: All students are protesters. All radicals are protesters. So, all students are radicals.

With the dictionary for Example 4 and one addition we get:

$(\forall x)(Sx \supset Px)$
$(\forall x)(Rx \supset Px)$
$\therefore (\forall x)(Sx \supset Rx)$

Rx = x is a radical

The tree is:

a \ 1. $(\forall x)(Sx \supset Px)$ P
a \ 2. $(\forall x)(Rx \supset Px)$ P
✓ 3. $\sim(\forall x)(Sx \supset Rx)$ NC
a ✓ 4. $(\exists x)\sim(Sx \supset Rx)$ 3, QN
5. $\sim(Sa \supset Ra)$ 4, EI
6. Sa } 5, PC
7. $\sim Ra$ }
✓ 8. $Sa \supset Pa$ 1, UI
✓ 9. $Ra \supset Pa$ 2, UI
10. $\sim Sa$ Pa 8, PC
X
11. $\sim Ra$ Pa 9, PC
↑

The tree will not close. A countermodel may be read off:

	S	P	R
a	1	1	0

This describes a world where a is a student protester but not a radical: clearly such a world is possible, so we have a counterexample.

The expansion of the argument for a shows the premises true and the conclusion false:

$Sa \supset Pa = 1$
$Ra \supset Pa = 1$
$Sa \supset Ra = 0$

All of these examples are quite simple and straight forward. But there are more difficult cases. Before considering these we specify Efficiency Rules and a revised Completion Rule.

Efficiency Rule 1: Don't branch until you have to.

Efficiency Rule 2: In general, follow the order

PC
QN
EI
UI ↓

The first rule is familiar from PC. The second rule advises us that we will usually save some work if we do things in the following order: first of all do any necessary PC resolving; then use QN to eliminate tildes from the front of formulae of the form $\sim(\exists \nu)\,\alpha$ or $\sim(\forall \nu)\,\alpha$; then apply EI; then UI.

In Example 6 we follow Rule 2 to test the following argument-form for validity; in 7 we do not follow the rule.

$$\frac{(\forall x)(Fx \supset Gx)}{\therefore\quad (\forall x)\, Fx \supset (\forall x)\, Gx}$$

Example 6:

a \	1.	$(\forall x)(Fx \supset Gx)$	P
√	2.	$\sim[(\forall x)\, Fx \supset (\forall x)\, Gx]$	NC
a \	3.	$(\forall x)\, Fx$	} 2, PC
√	4.	$\sim(\forall x)\, Gx$	
a √	5.	$(\exists x) \sim Gx$	4, QN
	6.	$\sim Ga$	5, EI
	7.	Fa	3, UI
√	8.	$Fa \supset Ga$	1, UI
	9.	$\sim Fa$ $\quad$ Ga	8, PC
		X $\quad$ X	

Example 7:

ba \	1.	$(\forall x)(Fx \supset Gx)$	P
√	2.	$\sim[(\forall x)\, Fx \supset (\forall x)\, Gx]$	NC
	3.	$Fa \supset Ga$	1, UI
ba \	4.	$(\forall x)\, Fx$	} 2, PC
√	5.	$\sim(\forall x)\, Gx$	
	6.	Fa	4, UI
	7.	$(\exists x) \sim Gx$	5, QN
b √	8.	$\sim Gb$	7, EI
√	9.	$Fb \supset Gb$	1, UI
	10.	Fb	4, UI
	11.	$\sim Fb$ $\quad$ Gb	9, PC
		X $\quad$ X	

Failure to do a PC resolution of 2, then QN, then EI, before the UI's of 1 and 4 lead to extra work. Notice also that we had to do UI twice, first to *a* then *b*. Failure to UI with *b* would have left the tree incomplete.

The completion rule for PC-trees has to be modified in one way to make it suitable for MQT. You will remember that when UI is applied to a formula the formula is not thereby finished with. UI may be applied repeatedly.

We now introduce the idea of the *Total UI of a formula in a path.*

> If a formula $(\forall \nu)\, \alpha$ has been universally instantiated at least once, and every formula of the form $(\exists\, \omega)\, \beta$ in the same path has been existentially instantiated, and $(\forall \nu)\alpha$ has been universally instantiated with respect to every individual constant occurring in the same path, then $(\forall \nu)\alpha$ *has been Totally Universally Instantiated,* **TUI***'d, in that path.*

Once a formula has been TUI'd in *every* open path whose end stems from it, we can show this by *crossing the backslash* as set out below in line 4:

a ✓	1.	$(\exists x)\, Fx$	P
b ✓	2.	$(\exists x)\, Gx$	P
✓	3.	$\sim(\exists x)(Fx \vee Gx)$	NC
ba X	4.	$(\forall x) \sim(Fx \vee Gx)$	3, QN
	5.	Fa	1, EI
	6.	Gb	2, EI
	7.	$\sim(Fa \vee Ga)$	4, UI
	8.	$\sim(Fb \vee Gb)$	4, UI

We leave it to the reader to complete the tree.

Completion Rule: Keep going until *either*

(i) All paths close, *or*

(ii) There is at least one open path in which every formula with a universal quantifier main operator is crossed, TUI, and in which every other unticked formula is free of quantifiers, dyadic propositional operators and double negations.

If (ii) occurs, the tree will never close, and a countermodel may be read off from one open path.

The importance of TUI can be seen in Example 8.

Example 8: To see if $(\exists x)(Fx \supset (\forall x)\, Fx)$ is an MQT-Necessity.

We could go only to line 7:

✓	1.	$\sim(\exists x)(Fx \supset (\forall x)\, Fx)$	NF
a \	2.	$(\forall x) \sim(Fx \supset (\forall x)\, Fx)$	1, QN
✓	3.	$\sim(Fa \supset (\forall x)\, Fx)$	2, UI
	4.	Fa	} 3, PC
✓	5.	$\sim(\forall x)\, Fx$	
b ✓	6.	$(\exists x) \sim Fx$	5, QN
	7.	$\sim Fb$	6, EI

Since the tree does not close, at this point we might be tempted to read off the following as a counterexample

	F
a	1
b	0

But the expansion of the formula being tested is:

$$(Fa \supset (Fa \,\&\, Fb)) \vee (Fb \supset (Fa \,\&\, Fb))$$

which is true in our world:

$$(1 \supset (1 \,\&\, 0)) \vee (0 \supset (1 \,\&\, 0))$$
$$= \ \ldots \vee 1$$
$$= \ 1$$

So we do *not* have a counterexample. But, we have not Totally UI'd the formula in line 2. If we do so, the tree continues as:

✓	8.	$\sim(Fb \supset (\forall x)\, Fx)$	2, UI
	9.	Fb	} 8, PC
	10.	$\sim(\forall x)\, Fx$	
		X	

So, $(\exists x)(Fx \supset (\forall x)\, Fx)$ is an MQT-Necessity.

EXERCISE 12.2

1. Complete the justifications in the following (correct) truth-trees.

(a)

√ 1. $\sim((\exists x)\,Fx\ .\supset.\ (\exists x)(Fx \lor Gx))$ NF
√ 2. $(\exists x)\,Fx$
√ 3. $\sim(\exists x)(Fx \lor Gx)$
\ 4. $(\forall x) \sim (Fx \lor Gx)$
5. Fa
√ 6. $\sim(Fa \lor Ga)$
7. $\sim Fa$
8. $\sim Ga$
X

(b)

√ 1. $\sim((\forall x)\,Fx\ .\supset.\ (\forall x)(Fx \,\&\, Gx))$ NF
\ 2. $(\forall x)Fx$
√ 3. $\sim(\forall x)(Fx \,\&\, Gx)$
√ 4. $(\exists x) \sim (Fx \,\&\, Gx)$
√ 5. $\sim(Fa \,\&\, Ga)$
6. $\sim Fa$ $\sim Ga$
7. Fa Fa
X

(c)

√ 1. $\sim((\exists x)\,Fx \supset ((\exists x)\,Gx \supset (\forall x)\,Fx))$ NF
√ 2. $(\exists x)\,Fx$
√ 3. $\sim((\exists x)\,Gx \supset (\forall x)\,Fx)$
√ 4. $(\exists x)\,Gx$
√ 5. $\sim(\forall x)\,Fx$
√ 6. $(\exists x) \sim Fx$
7. Fa
8. Gb
9. $\sim Fc$

(d)

√ 1. $\sim((\forall x)(Fx \supset Gx) \supset ((\forall x)\,Fx \supset (\forall x)\,Gx))$ NF
\ 2. $(\forall x)(Fx \supset Gx)$
√ 3. $\sim((\forall x)\,Fx \supset (\forall x)\,Gx)$
\ 4. $(\forall x)\,Fx$
√ 5. $\sim(\forall x)\,Gx$
√ 6. $(\exists x) \sim Gx$
7. $\sim Ga$
√ 8. $Fa \supset Ga$
9. Fa
10. $\sim Fa$ Ga
X X

(e)

√ 1. $\sim((\forall x)(p \supset Fx) \supset (p \supset (\forall x)\,Fx))$ NF
\ 2. $(\forall x)(p \supset Fx)$
√ 3. $\sim(p \supset (\forall x)\,Fx)$
4. p
√ 5. $\sim(\forall x)\,Fx$
√ 6. $(\exists x) \sim Fx$
7. $\sim Fa$
√ 8. $p \supset Fa$

9. $\sim p$ Fa
× ×

(f) 1. Fa P
✓ 2. $\sim(\exists x)Fx$ NC
\ 3. $(\forall x)\sim Fx$
4. $\sim Fa$
×

(g) \ 1. $(\forall x)(Mx \supset Px)$ P
\ 2. $(\forall x)(Sx \supset Mx)$ P
✓ 3. $\sim(\forall x)(Sx \supset Px)$ NC
✓ 4. $(\exists x) \sim (Sx \supset Px)$
✓ 5. $\sim(Sa \supset Pa)$
6. Sa
7. $\sim Pa$
✓ 8. $(Ma \supset Pa)$
9. $\sim Ma$ Pa
×
✓10. $(Sa \supset Ma)$
11. $\sim Sa$ Ma
× ×

(h) ✓ 1. $(\exists x)(Fx \supset Gx)$ P
✓ 2. $\sim((\forall x)Fx \supset (\exists x)Gx)$ NC
\ 3. $(\forall x)Fx$
✓ 4. $\sim(\exists x)Gx$
\ 5. $(\forall x)\sim Gx$
✓ 6. $Fa \supset Ga$
7. Fa
8. $\sim Ga$
9. $\sim Fa$ Ga
× ×

(i) \ 1. $(\forall x)(Fx \supset (Gx \vee Bx))$ P
✓ 2. $(\exists x)\sim(Fx \supset Gx)$ P
✓ 3. $\sim(\exists x)(Bx \,\&\, Fx)$ NC
\ 4. $(\forall x)\sim(Bx \,\&\, Fx)$
✓ 5. $\sim(Fa \supset Ga)$
6. Fa
7. $\sim Ga$

✓ 8. $\sim(Ba \,\&\, Fa)$

9. $\sim Ba$ $\qquad$ $\sim Fa$ ×

✓ 10. $Fa \supset (Ga \vee Ba)$

11. $\sim Fa$ × $\qquad$ ✓ $Ga \vee Ba$

12. Ga × $\qquad$ Ba ×

(j) \ 1. $(\forall x)(Fx \supset Gx)$ P
\ 2. $(\forall x)(Hx \supset Ix)$ P
✓ 3. $\sim[(\forall x)(Gx \supset Hx) \supset (\forall x)(Fx \supset Ix)]$ NC
\ 4. $(\forall x)(Gx \supset Hx)$
✓ 5. $\sim(\forall x)(Fx \supset Ix)$
✓ 6. $(\exists x)\sim(Fx \supset Ix)$
✓ 7. $\sim(Fa \supset Ia)$
8. Fa
9. $\sim Ia$
✓ 10. $Ga \supset Ha$
✓ 11. $Fa \supset Ga$
✓ 12. $Ha \supset Ia$

13. $\sim Fa$ × $\qquad$ Ga

14. $\sim Ha$ $\qquad$ Ia ×

15. $\sim Ga$ × $\qquad$ Ha ×

2. The tree below is used to test the argument-form

$$(\forall x)(p \supset Fx) \;\; / \therefore (p \supset (\forall x)Fx).$$

The steps shown are correct but not all the justifications (i.e. quoted lines and principles) are correct. For each line, state whether the justification given is correct or not. Which lines require ticks?

a \ 1. $(\forall x)(p \supset Fx)$ P
2. $\sim(p \supset (\forall x)Fx)$ NC
3. p 1, UI

4.	$\sim(\forall x)Fx$	2, QN
5.	$(\exists x)\sim Fx$	4, QN
6.	$\sim Fa$	4, UI
7.	$p \supset Fa$	1, EI
8.	$\sim p \qquad Fa$	7, PC
	× $\qquad$ ×	

3. Use MQT-Trees to show that the following are MQT-Necessities.
   - (a) $(\forall x)\,Fx \supset (\exists x)\,Fx$
   - (b) $(\exists x)\,Fx \supset (\exists x)(Fx \vee Gx)$
   - (c) $(\exists x)\,Fx \equiv (\exists y)\,Fy$
   - (d) $(\forall x)\,Fx \supset (\forall y)\,Fy$
   - (e) $(\exists x)(\forall y)(Fx \supset Gy) \equiv (\forall y)(\exists x)(Fx \supset Gy)$
   - (f) $(\forall x)(Fx \,\&\, Gx) \supset ((\forall x)\,Fx \,\&\, (\forall x)\,Gx)$
   - (g) $(\exists x)(Fx \vee Gx\,) \supset ((\exists x)\,Fx \vee (\exists x)\,Gx)$
   - (h) $(\forall x)(Fx \supset Gx) \supset \sim(\exists x)(Fx \,\&\, \sim Gx)$
   - (i) $(\forall x)[(\forall x)\,Fx \supset Fx]$
   - (j) $(\exists x)[(\exists x)\,Fx \supset Fx]$
   - (k) $((\exists x)\,Fx \,\&\, p) \supset (\exists x)(Fx \,\&\, p)$
   - (l) $(\exists x) \sim Gx \supset \sim(\forall x)(Fx \,\&\, Gx)$
   - (m) $((\forall x)\,Fx \vee (\forall x)\,Gx) \supset (\forall x)(Fx \vee Gx)$
   - (n) $(\forall x)(Fx \equiv p) \equiv ((\exists x)\,Fx \supset p) \,\&\, (p \supset (\forall x)\,Fx))$
   - (o) $Gx \supset Gx$

4. Use MQT-Trees to show that the following are MQT-Contradictions.
   - (a) $(\forall x)(Fx \,\&\, \sim Fx)$
   - (b) $(\exists x)(Fx \,\&\, Gx) \,\&\, (\forall x)(Fx \supset \sim Gx)$
   - (c) $(\exists x)(Fx \supset p) \equiv ((\forall x)\,Fx \,\&\, \sim p)$

5. Use MQT-Trees to show that the following argument-forms are valid.
   - (a) $(\exists x)Fx \;/\; \therefore\; (\exists x)(Fx \vee Gx)$
   - (b) $(\forall x)(Fx \,\&\, Gx) \;/\; \therefore\; (\forall x)Fx$
   - (c) $(\forall x)(Fx \supset Gx), (\exists x)\,Fx \;/\; \therefore\; (\exists x)\,Gx$
   - (d) $(\forall x)(Fx \supset Gx), (\exists x)\,Fx \;/\; \therefore\; (\exists x)(Gx \,\&\, Fx)$
   - (e) $(\forall x)(Fx \supset \sim Gx), (\exists x)\,Fx \;/\; \therefore\; (\exists x)(Fx \,\&\, \sim Gx)$
   - (f) $(\forall x)[Fx \supset (Gx \vee Hx)], (\forall x)(Gx \supset \sim Hx), (\forall x)(Hx \supset \sim Gx), (\exists x)\,Fx \;/\; \therefore$ $(\exists x)(Gx \not\equiv Hx)$
   - (g) $(\exists x)(Fx \equiv Gx) \;/\; \therefore\; (\exists x)(\exists y)(Fx \equiv Gy)$
   - (h) $(\forall x)[Fx \supset (Sx \vee Px)], \sim(\exists x)[Gx \,\&\, (Px \vee Sx)] \;/\; \therefore\; (\forall x)(Gx \supset \sim Fx)$

6. Symbolize the following arguments, using the dictionary provided, and test them for validity. If any is invalid provide a counterexample and show that the counterexample is indeed a counterexample.
   - (a) Not all members of parliament are elected. This is because some Senators are appointed by State Governments. Of course, Senators are members of parliament, and if a Senator is appointed then he is not elected.
     ($Mx = x$ is a member of parliament; $Ex = x$ is elected; $Sx = x$ is a Senator; $Ax = x$ is appointed by a State Government.)

(b) Aristotle was a philosopher, and Aristotle was a logician. So some logicians have been philosophers.
($Px = x$ was a philosopher; $Lx = x$ was a logician; $a =$ Aristotle.)

(c) Any number is either odd or even if it is a natural number. If a number is half a prime number, then it is neither odd nor even. Thus, if it is a number, half a prime is not a natural number.
($Nx = x$ is a number; $Ex = x$ is even; $Ox = x$ is odd; $Tx = x$ is a natural (number); $Hx = x$ is half a prime number.)

(d) Anyone with any public spiritedness will join in either giving money or time to flood relief. Since Jack has not given any time to flood relief, because he is not able to, and since he is public spirited we can conclude that he will give money to flood relief.
($Px = x$ is public spirited; $Gx = x$ will join in giving money to flood relief; $Tx = x$ will join in giving time to flood relief; $Ax = x$ is able to give time to flood relief; $j =$ Jack.)

(e) Some of the present increases in the minimum wage will cause prices to rise. Excessive increases in wages cause prices to rise. So it follows that some increases in the minimum wage at present are excessive.
($Ix = x$ is a present increase in the minimum wage; $Rx = x$ will cause prices to rise; $Ex = x$ is an excessive increase in wages.)

(f) Moderates are never protesters, but reformers are never moderates, hence reformers are protesters.
($Mx = x$ is a moderate; $Px = x$ is a protester; $Rx = x$ is a reformer.)

(g) Students who are good scholars are hard workers. Anyone who works hard learns to persevere. Since there are some students who do not learn to persevere, it follows that there are some students who are not good scholars.
($Sx = x$ is a student; $Lx = x$ is a good scholar; $Wx = x$ is a hard worker; $Px = x$ learns to persevere.)

(h) Only fools look for gold in this place. Anyone who looks for gold in this place is engaged in a fruitless search. Since there are fools in the world, it follows that some of them will engage in a fruitless search.
($Fx = x$ is a fool; $Lx = x$ looks for gold in this place; $Ex = x$ is engaged in a fruitless search; $Ix = x$ is in the world.)

(i) Teenagers over fifteen are able to leave school and work. Anyone who can leave school and work can earn large sums of money nowadays. So some of the people able to earn large sums of money these days are teenagers over fifteen.
($Fx = x$ is a teenager over fifteen; $Lx = x$ is able to leave school; $Wx = x$ is able to work; $Ex = x$ is able to earn large sums of money.)

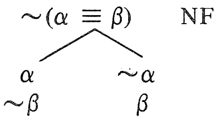
(j) None but the brave deserve the fair, and none but the willing are brave. So, none but the willing deserve the fair.
($Bx = x$ is brave; $Dx = x$ deserves the fair; $Wx = x$ is willing.)

12.3 TESTING MODAL RELATIONS

We may use MQT-trees to test for formal MQT relations between formulae. We use the tests as set out below:

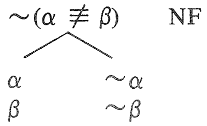
Relation	Test for MQT-Necessity
α *MQT-Implies* β	$\alpha \supset \beta$
α *is MQT- Equivalent to* β	$\alpha \equiv \beta$
α *is MQT-Contradictory to* β	$\alpha \not\equiv \beta$
α *is MQT-Contrary to* β	$\alpha \supset \sim\beta$
α *is MQT-Sub-contrary to* β	$\sim\alpha \supset \beta$

In fact the first two can be amalgamated into one test in trees, and the last three can also be amalgamated into one test. When we test α *is MQT-Equivalent to* β the tree begins as follows:



If the left path *alone* closes then α *MQT-Implies* β. If the right path *alone* closes then β *MQT-Implies* α. If *both* close then α *is MQT-Equivalent to* β.

When we test α *is MQT-Contradictory to* β the tree begins as follows:



If the left path *alone* closes then α *is MQT-Contrary to* β because both cannot be true, but they can both be false. If the right path *alone* closes then α *is MQT-Sub-contrary to* β, because both cannot be false, but they can both be true. If *both* paths close, then α *is MQT-Contradictory to* β.

These tests can be extended from the formal MQT relations to testing modal relations between propositions. The extension is accomplished by using possible-truth trees and the general definitions of necessary implication, necessary equivalence, contradictoriness, contrariety and sub-contrariety set out in §3.7.

EXERCISE 12.3

1. Use MQT-Trees to verify that the formal relations set out in the two squares of opposition in §11.2 do hold.

2. Test the following pairs of formulae for the relation indicated.

   (a) $(\forall x)(Fx \supset Gx)$ is MQT-*Contrary* to $(\forall x)(Fx \supset \sim Gx)$
   (b) $(\exists x)(Fx \,\&\, Gx)$ is MQT-*Sub-contrary* to $(\exists x)(Fx \,\&\, \sim Gx)$
   (c) $(\forall x)(Fx \,\&\, Gx)$ MQT-*Implies* $(\forall x)\, Fx$
   (d) $(\forall x)(Fx \,\&\, p)$ is MQT-*Equivalent* to $(\forall x)\, Fx \,\&\, p$
   (e) $(\exists x)(Fx \vee p)$ is MQT-*Equivalent* to $(\exists x)\, Fx \vee p$

12.4 FURTHER EQUIVALENCES AND MINIMUM SCOPE

We have already met the four quantifier negation equivalences. We are now going to consider a group of equivalences, of which the quantifier negation equivalences are part, which are useful for putting formulae into *Minimum Scope Form* (MSF), or into *Prenex Normal Form* (PNF).

We begin by listing the four QN equivalences. Where α is a wff of MQT:

$$\sim(\forall x)\,\alpha \quad\equiv\quad (\exists x)\sim\alpha \tag{1}$$
$$\sim(\exists x)\,\alpha \quad\equiv\quad (\forall x)\sim\alpha \tag{2}$$
$$\sim(\exists x)\sim\alpha \quad\equiv\quad (\forall x)\,\alpha \tag{3}$$
$$\sim(\forall x)\sim\alpha \quad\equiv\quad (\exists x)\,\alpha \tag{4}$$

Our next equivalence is called the *change of Bound Variable* principle (CBV). Examples of it are (5) and (6)

$$(\exists x)\,Fx \quad\equiv\quad (\exists y)\,Fy \tag{5}$$
$$(\forall x)\,Fx \quad\equiv\quad (\forall y)\,Fy \tag{6}$$

The general idea of CBV is that all bound occurrences of a variable in a wff, where the binding is by the one quantifier, can be replaced by some other variable, so long as that new variable will not change the pattern of binding in the formula. For example, in (7) we display the binding pattern.

$$(\exists x)(Fx \,\&\, Fy) \tag{7}$$

We can replace x with z to get (8), and the binding pattern remains the same.

$$(\exists z)(Fz \,\&\, Fy) \tag{8}$$

We cannot replace x in (7) with y to get (9), because the binding pattern alters.

$$(\exists y)(Fy \,\&\, Fy) \tag{9}$$

(7) is equivalent to (8).

$$(\exists x)(Fx \,\&\, Fy) \quad\equiv\quad (\exists z)(Fz \,\&\, Fy) \tag{10}$$

(10) is an MQT-Necessity.

To set out the general principle, CBV, we introduce the following symbolism:

Let: α be any wff of MQT
ν and ω be any individual variables
Q be either $\exists$ or $\forall$.

$\alpha(\omega//\nu)$ means: the result of substituting ω for every occurrence of ν in α

Now we can set out the principle CBV:

$$(\mathrm{Q}\nu)\,\alpha \;\equiv\; (\mathrm{Q}\,\omega)\,\alpha(\omega//\nu)$$

provided ω does not occur in α. (11)

We now set out some change of scope examples (12) to (19). In these formulae we allow p to represent any wff in which x does not occur free.

$$(\exists x)(Fx \,\&\, p) \;\equiv.\; (\exists x)\,Fx \,\&\, p \tag{12}$$
$$(\exists x)(Fx \vee p) \;\equiv.\; (\exists x)\,Fx \vee p \tag{13}$$
$$(\forall x)(Fx \,\&\, p) \;\equiv.\; (\forall x)\,Fx \,\&\, p \tag{14}$$
$$(\forall x)(Fx \vee p) \;\equiv.\; (\forall x)\,Fx \vee p \tag{15}$$

$$(\exists x)(p \,\&\, Fx) \;\equiv.\; p \,\&\, (\exists x)\,Fx \tag{16}$$
$$(\exists x)(p \vee Fx) \;\equiv.\; p \vee (\exists x)\,Fx \tag{17}$$

$$(\forall x)(p \mathbin{\&} Fx) \equiv .\ p \mathbin{\&} (\forall x)\, Fx \tag{18}$$
$$(\forall x)(p \vee Fx) \equiv .\ p \vee (\forall x)\, Fx \tag{19}$$

In fact, using (12) to (15), we can show (16) to (19). For example:

$$\begin{array}{lll}
 & (\exists x)(p \mathbin{\&} Fx) & \\
\Leftrightarrow & (\exists x)(Fx \mathbin{\&} p) & \text{Com} \\
\Leftrightarrow & (\exists x)\, Fx \mathbin{\&} p & (12) \\
\Leftrightarrow & p \mathbin{\&} (\exists x)\, Fx & \text{Com}
\end{array}$$

So $(\exists x)(p \mathbin{\&} Fx) \equiv .\ p \mathbin{\&} (\exists x)\, Fx$

In each of (12) to (19) we assume that there is no free occurrence of x in p. In the LHE the scope of the quantification is maximum, so the quantification is the main operator in the LHE. In the RHE the scope is minimum, and the main operator is not the same as in the LHE.

The scope of a quantification can be changed over & and ∨ so long as there is no binding pattern change. We now use such equivalences and the principle of replacement of logical equivalents to see what happens when hook is involved.

$$\begin{array}{lll}
 & (\exists x)(Fx \supset p) & \\
\Leftrightarrow & (\exists x)(\sim Fx \vee p) & \text{MI} \\
\Leftrightarrow & (\exists x) \sim Fx \vee p & (13) \\
\Leftrightarrow & \sim (\forall x)\, Fx \vee p & \text{QN} \\
\Leftrightarrow & (\forall x)\, Fx \supset p & \text{MI}
\end{array}$$

So we get (20)

$$(\exists x)(Fx \supset p) \equiv .\ (\forall x)\, Fx \supset p \tag{20}$$

Similarly (21)

$$(\forall x)(Fx \supset p) \equiv .\ (\exists x)\, Fx \supset p \tag{21}$$

Consider:

$$\begin{array}{lll}
 & (\exists x)(p \supset Fx) & \\
\Leftrightarrow & (\exists x)(\sim p \vee Fx) & \text{MI} \\
\Leftrightarrow & \sim p \vee (\exists x)\, Fx & (17) \\
\Leftrightarrow & p \supset (\exists x)\, Fx & \text{MI}
\end{array}$$

So we get (22), and similarly (23):

$$(\exists x)(p \supset Fx) \equiv .\ p \supset (\exists x)\, Fx \tag{22}$$
$$(\forall x)(p \supset Fx) \equiv .\ p \supset (\forall x)\, Fx \tag{23}$$

These principles can be used to put a formula into MSF. When a formula, α, is put into MSF we are finding a closed wff, β, such that α is logically equivalent to β, and in β no quantifier, propositional letter, individual constant, or free occurrence of an individual variable occurs inside the scope of any quantifier. For example:

$$(\forall x)(Fx \supset (p \mathbin{\&} (\exists y)(Gy \mathbin{\&} Ha))) \tag{24}$$

$$\begin{array}{lll}
\Leftrightarrow & (\exists x)\, Fx \supset (p \mathbin{\&} (\exists y)(Gy \mathbin{\&} Ha)) & \text{by } (21) \\
\Leftrightarrow & (\exists x)\, Fx \supset (p \mathbin{\&} ((\exists y)\, Gy \mathbin{\&} Ha)) & \text{by } (12)
\end{array}$$

This last formula is an MSF of (24).

Similarly, we can use these principles to put a formula into PNF. To do this we find a formula, β, such that the formula is logically equivalent to β, and β is closed, contains no vacuous quantifiers, and is of the form

$$(\mathrm{Q}\nu)\ \ldots\ (\mathrm{Q}\omega)\, \alpha$$

and α is quantifier free. For example we begin with (24)

$\Leftrightarrow \quad (\forall x)(Fx \supset (\exists y)(p \,\&\, (Gy \,\&\, Ha)))$ by (16)
$\Leftrightarrow \quad (\forall x)(\exists y)(Fx \supset (p \,\&\, (Gy \,\&\, Ha)))$ by (22)

So we have a PNF of (24). PNF is important for the application of the short cuts set out in §11.3 and §11.4.

EXERCISE 12.4

1. Use truth-trees to verify all the specific equivalences mentioned in this section.

2. Use the principle of substitution of logical equivalents to find an MSF formula equivalent to each of the following

   (a) $Fx \supset p$
   (b) $(\forall x)(\forall y)(Fx \supset Gy)$
   (c) $(\forall x)(\exists y)(Fx \,\&\, Gy)$
   (d) $(\forall x)(p \supset (\exists y)(Fy \vee Gx))$
   (e) $(\forall x)(Fx \supset (\exists y)\, Gy)$
   (f) $(\exists x)((Ga \,\&\, Fa) \supset (Gx \,\&\, Fx))$
   (g) $(\exists x)((Fx \supset Gx) \supset (Fa \supset Ga))$
   (h) $p \vee q$
   (i) $(Fx \,\&\, Fa) \supset (\forall x)(Fx \,\&\, Fa)$
   (j) $(\forall x)(\forall y)((p \,\&\, Ga) \supset (Gz \vee (Gx \,\&\, Fy)))$

3. Use your knowledge and intuitions to say whether or not the following pairs of formulae are equivalent. If doubt persists use truth-trees.

   (a) $\sim(\forall y)\, Fy$ and $(\exists y) \sim Fy$
   (b) $(\exists x)\, Gx$ and $\sim(\exists x) \sim Gx$
   (c) $(\exists x)\, Hx$ and $(\exists y)\, Hy$
   (d) Hx and $(\forall y)\, Hy$
   (e) Gy and $\sim(\exists x) \sim Gx$
   (f) $(\exists y)(Fy \,\&\, Gy)$ and $(\exists y)\, Fy \,\&\, (\exists y)\, Gy$
   (g) $(\forall z)(Gz \vee Hz)$ and $(\forall z)\, Gz \vee (\forall z)\, Hz$
   (h) $(\exists y)(Fy \supset Hy)$ and $\sim(\forall y)(Fy \,\&\, \sim Hy)$
   (i) $(\forall y)(Fy \,\&\, Gy)$ and $(\forall y)\, Fy \,\&\, (\forall y)\, Gy$
   (j) $(\forall y)(Fy \,\&\, Gx)$ and $(\forall y)(Fy \,\&\, Gy)$

4. Use the principle of substitution of logical equivalents to find a PNF formula equivalent to each of the following.

   (a) $(\exists x)\, Fx \supset p$
   (b) $(\forall x)(Fx \supset (\exists y)\, Gy)$
   (c) $(\forall x)((\exists y)\, Gy \supset Fx)$
   (d) $(\forall x)(Fx \,\&\, (\exists x)\, Gx)$
   (e) $(\exists x)\, Fx \vee (\forall x)\, Gx$
   * (f) $(\forall x) \sim Fx \equiv \sim(\exists y)\, Fy$

Puzzle 12 The flight crew of a jumbo jet includes a pilot (P), co-pilot (C) and navigator (N), whose names are brown (b), Jones (j) and Smith (s), not necessarily in that order. Travelling on board are three passengers: Mr Brown (mb), Mr Jones (mj) and Mr Smith (ms). The following facts are known.

1. The co-pilot lives in New South Wales (N).
2. Mr Brown thinks a yogi is a type of bear.
3. Mr Smith lives in Queensland (Q).
4. Jones borrowed a Rubik Snake from the navigator.
5. The passenger whose surname is the same as the co-pilot's lives in Victoria (V).
6. The pilot's favourite meal is red herring.
7. The co-pilot lives next door to one of the passengers, an advanced student of yoga.

Who is the pilot? (The following grids may help.)

	P	C	N
b			
j			
s			

	N	Q	V
mb			
mj			
ms			

12.5 SUMMARY

PC-trees (Ch. 6) may be augmented to become *MQT-trees,* which provide a decision procedure for MQT. The *Replacement Rules* are as for PC, plus *Quantifier Negation (QN)* and *Existential Instantiation (EI).* In specifying the rules for EI and *Universal Instantiation (UI),* $\phi\nu$ denotes the scope of the initial quantification, and $\phi\kappa$ is an itemization of $\phi\nu$ to the individual constant κ.

QN:

$$\begin{array}{l} \surd \sim(\exists\nu)\,\alpha \\ \quad(\forall\nu) \sim\alpha \end{array} \qquad\qquad \begin{array}{l} \surd \sim(\forall\nu)\,\alpha \\ \quad(\exists\nu) \sim\alpha \end{array}$$

EI:

$$\begin{array}{l} \kappa\ \surd\ (\exists\nu)\,\phi\nu \\ \qquad\qquad\ \phi\kappa \qquad (\kappa\ \textit{new} \text{ to the path}) \end{array}$$

UI:

$$\begin{array}{l} \kappa \ldots \backslash \quad (\forall\nu)\,\phi\nu \\ \qquad\qquad\quad \phi\kappa \qquad (\text{for } \textit{any}\ \kappa) \end{array}$$

Existential expressions may be instantiated once only, to a new constant. UI is not a replacement rule, and may be used as often as desired. With UI, it is usually best to instantiate to *old* constants as much as possible. Never instantiate through a $\sim$: use QN to shift it. In multiply quantified expressions, instantiate from left to right.

Efficiency Rules: Don't branch until you have to.
In general, follow the order

$$\left.\begin{array}{l} \text{PC} \\ \text{QN} \\ \text{EI} \\ \text{UI} \end{array}\right\downarrow$$

Closure Rule: Same as for PC.

A formula $(\forall \nu)\alpha$ has been *Totally Universally Instantiated (TUI'd) in a path* if it has been UI'd at least once, it has been UI'd with respect to every IC in the same path, and every formula of the form $(\exists\, \omega)\alpha$ in the same path has been EI'd. To indicate that a formula has been TUI'd in *every* open path whose end stems from it we *cross* the backslash in front of the formula: $\cancel{\backslash}$ $(\forall \nu)\ \alpha$.

Completion Rule: Keep going until *either*

(i) all paths close *or*

(ii) there is at least one open path in which every formula with a universal quantifier main operator is crossed, and in which every unticked formula is free of quantifiers, dyadic propositional operators and double negations.

If alternative (ii) occurs, the path is permanently open, and a countermodel may be read from it.

The overall MQT-tree tests for propositions and arguments are the same as the PC-tree tests. Various MQT-modal relations may be defined and tested by MQT-trees; the standard modal relations may be tested by extending the possible-truth tree method to MQT-trees (see § 12.3).

Let α be any MQL wff, ν and ω be IVs, Q be either $\forall$ or $\exists$, and $\alpha(\omega//\nu)$ denote the result of substituting ω for every occurrence of ν in α. Then the *Change of Bound Variable (CBV)* equivalence principle is as follows.

CBV: If ω does not occur in α then $(Q\nu)\,\alpha \quad \Leftrightarrow \quad (Q\,\omega)\ \alpha\,(\omega//\nu)$

e.g., $(\forall x)\ Fx \Leftrightarrow (\forall y)\ Fy$ and $(\exists x)\ Fx \Leftrightarrow (\exists y)\ Fy$

Let Q be $\forall$ or $\exists$, and p be any formula with no free occurrence of x. Then the following *Change of Scope (CS)* equivalences apply.

CS:

$(Qx)\ (Fx \,\&\, p) \Leftrightarrow (Qx)\ Fx \,\&\, p$

$(Qx)\ (Fx \vee p) \Leftrightarrow (Qx)\ Fx \vee p$

$(Qx)\ (p \supset Fx) \Leftrightarrow p \supset (Qx)\ Fx$

$(\forall x)\ (Fx \supset p) \Leftrightarrow (\exists x)\ Fx \supset p$

$(\exists x)\ (Fx \supset p) \Leftrightarrow (\forall x)\ Fx \supset p$

QN, CBV and CS equivalences may be used to put a formula into *Minimum Scope Form (MSF)*, where no quantifiers, propositional letters, ICs or free IVs occur in the scope of any quantifier. Formulae in MSF are closed. These equivalences may also be used to put a formula into *Prenex Normal Form (PNF)*. A formula is in PNF iff it is closed, has no vacuous quantifiers, and is of the form $(Q\nu)\ \ldots\ (Q\,\omega)\alpha$ where α is quantifier free.

13 Logic Diagrams

13.1 INTRODUCTION

Logic diagrams are geometrical figures used to represent either terms, sets, propositions or forms, with a view to solving logical problems. Leonhard Euler (1707-1783) was the first to make systematic use of diagrams in logic. Earlier logicians, such as Leibniz, had made some use of diagrams, but Euler set out a detailed system, usually referred to as *Euler Diagrams.* A contemporary of Euler, J. H. Lambert, also set out a system somewhat like Euler's, but whereas Euler used circles, Lambert used lines. The systems of Euler and Lambert were, in some sense, displaced by *Venn Diagrams.* These diagrams were invented by John Venn (1834-1923) and are in common use today. Venn diagrams generally use circles and ellipses to represent either terms or propositions.

In 1881, Alan Marquand introduced a rectangularised style of Venn Diagram which promised greater efficiency in handling more complex propositions and terms. Various logicians, Lewis Carroll, M. Karnaugh and E. W. Veitch, in particular, have further developed the rectangular diagrams. Various other diagrams have been developed, even one consisting of cubes, but our main concern will be with Venn diagrams and Veith-Karnaugh maps. We will also look briefly at Euler diagrams. Carroll diagrams are very much like Veitch-Karnaugh maps ("Karnaugh maps" for short).

Our principal interest will be in the use of diagrams to represent *forms*. The diagrams which we look at are of two kinds. There are the *categorical* diagrams and the *fill-in* diagrams. The difference is considerable and will become clearer as we go along. Euler diagrams are categorical, whereas Venn diagrams and Karnaugh maps are fill-in diagrams. From a practical point of view the difference can be seen in terms of *drawing* and *filling-in*. Categorical diagrams are simply drawn, whilst fill-in diagrams are drawn and then filled in.

13.2 REPRESENTING FORMULAE

Euler diagrams, Venn diagrams and Karnaugh maps were used traditionally to represent **A E I** and **O** forms. We are going to show how to use such diagrams to represent formulae of MQT. We begin with the fill-in diagrams.

Fill-In Diagrams:

To use a fill-in diagram we must first draw a suitable blank diagram, and then fill it in to represent a formula. Set out below are formulae and blank diagrams. For each formula there is both a blank Venn Diagram and a blank Karnaugh Map. There are two things to note. First, in each diagram there is an area for each predicate in the related formula and an area for the complement of each predicate. For example, in the first pair of diagrams there is, in each diagram, an area for F, marked F, and an area for *non-F*, marked F'. Secondly, in each of the second and third pairs of diagrams we have numbered the areas for ease of reference. But normally the areas are not numbered.

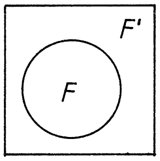


Fig 1

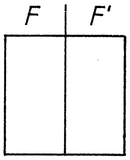


Fig. 2

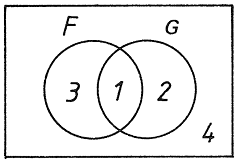


Fig 3

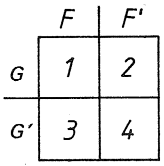


Fig. 4

In Figures 3 and 4 area 1 is for all those items which are both F and G; area 2 is for all which are *non-F* and G; area 3 is for all which are F and *non-G*; area 4 is for all which are *non-F* and *non-G*.

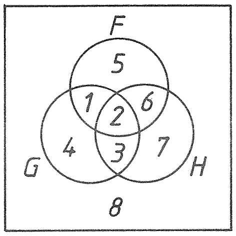


Fig. 5

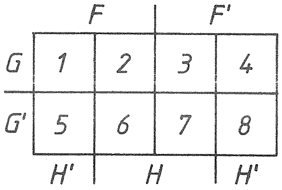


Fig. 6

In Figures 5 and 6 the areas are as follows:
area 1: items *F, G, non-H*; area 2: items *F, G, H*; area 3: items *non-F, G* and *H*; area 4: items *non-F, G, non-H*; area 5: items *F, non-G, non-H*; area 6: items *F, non-G, H*; area 7: items *non-F, non-G, H*; area 8: items *non-F, non-G, non-H.*

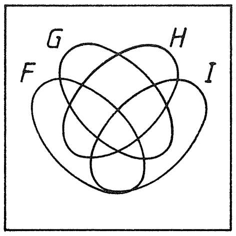


Fig. 7

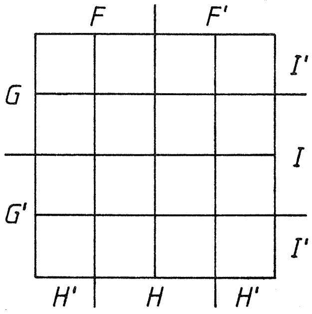


Fig. 8

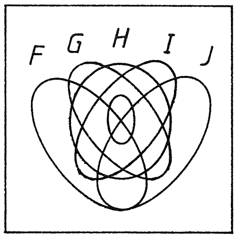


Fig. 9

Figures 7 and 8 deal with four predicates, and Figure 9 is a Venn diagram for five predicates.

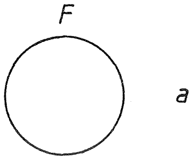
Note that in the Venn diagrams the area outside the circles is one of the areas taken into account. Usually, Venn diagrams are drawn without the surrounding rectangle or square. But the area outside the circles must be taken into account. Carroll diagrams for one or two predicates would be the same as the Karnaugh maps.

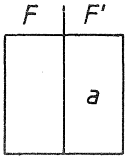
The number of areas required in a blank diagram is 2^n where n is the number of monadic predicate letters:

No. of Predicates:	1	2	3	4	... n
No. of Areas:	2	4	8	16	... 2^n

So much for blank diagrams. Now we must discover how to *fill them in* to represent formulae. There are *five devices* used. We now set out each of these devices in turn, with a formula and maps filled-in to represent the formula. Initially we will restrict ourselves to one-predicate formulae, and so fill-in Figures 1 and 2.

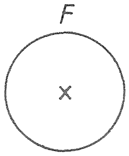
(i) *individual constants* are placed in the area of what is predicated of them.

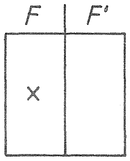




$\sim Fa$

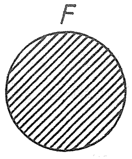
(ii) a *cross* is placed in an area in which there is at least one item.

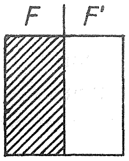




$(\exists x)\ Fx$

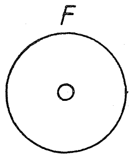
(iii) areas in which there are no items are *shaded out*. Such areas are *null areas*.

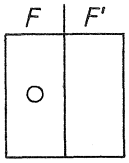




$\sim(\exists x)\ Fx$

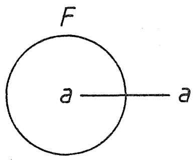
(iv) Null areas may be shown with a *small circle* instead of shading.

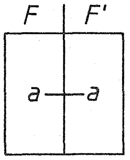




$(\forall x) \sim Fx$

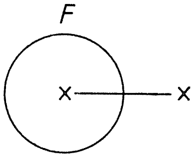
(v) a *disjunction bar* is used to represent alternatives.

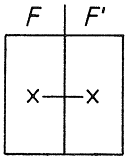




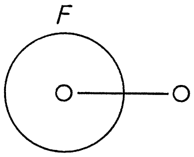
$Fa \vee \sim Fa$

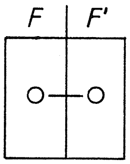
The disjunction bar is combined with the cross and the circle below.





$(\exists x)(Fx \vee {\sim} Fx)$

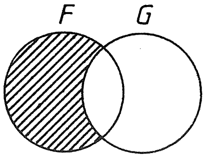


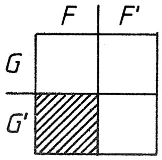


${\sim}(\exists x)\, Fx \vee {\sim}(\exists x)\, {\sim} Fx$

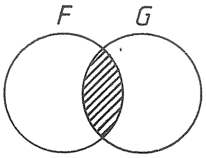
We now turn to some formulae containing two predicate letters. Without further ado we set out the diagrams, both Venn and Karnaugh, which result from filling in Figures 3 and 4 for the **A E I** and **O** forms.

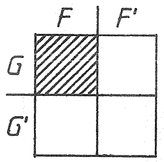
A $(\forall x)(Fx \supset Gx)$ *Every F is G*



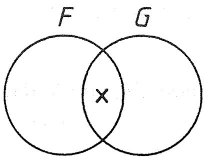


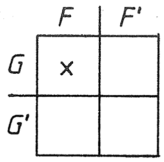
E $(\forall x)(Fx \supset {\sim} Gx)$ *No F is G*



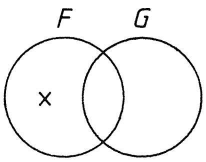


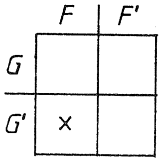
I $(\exists x)(Fx \,\&\, Gx)$ *Some F is G*



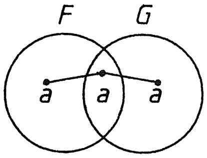


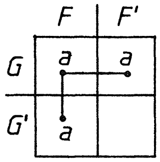
O $(\exists x)(Fx \,\&\, {\sim} Gx)$ *Some F is not G*





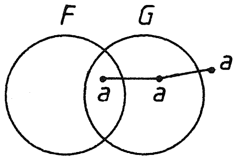
Note how, in a sense, the **A** form is represented by filling-in the diagram for *Nothing is F and non-G*. Now look carefully at the following two tricky cases, and work out the explanation for each.

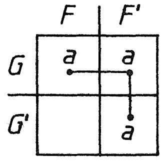




$Fa \vee Ga$

Each disjunction bar has three *points of disjunction*. To make this clear in the Venn diagram the bar is kinked. Remember that $p \supset q \Leftrightarrow \sim p \vee q$.



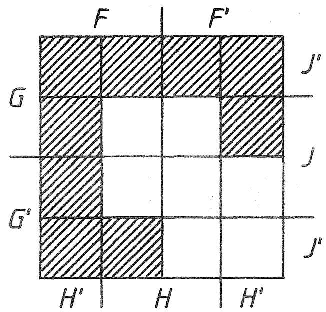


$Fa \supset Ga$

Finally we represent the following formula in a Karnaugh map.

$(\forall x)((Fx \vee Gx) \supset (Hx \& Jx))$

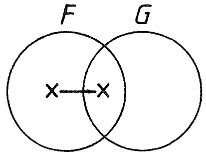
We need a sixteen-area map because there are 4 monadic predicates, and we fill it in to get

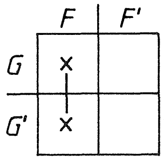


The diagram shows that no area which is F or G outside of the area which is both H and J has any members.

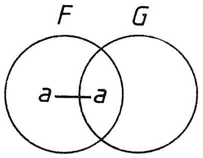
Two things must be noted. The first is that these diagrams represent the particular forms no matter how many items are in the world. An empty area is empty no matter how large the world. A cross does not limit the membership of an area to one item, it only says that *at least* one item is in that area.

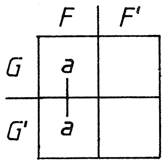
The second thing is that a form may be represented in a diagram which has *more areas* than are needed for that form alone. So we can represent $(\exists x)\ Fx$ in Figures 3 or 4 to get:





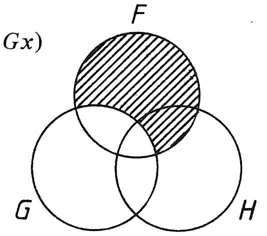
We need the disjunction bar because $(\exists x)\ Fx$ does not say whether the at least one F is G or non-G. Similarly for Fa we get:

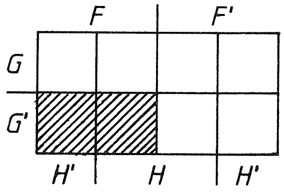




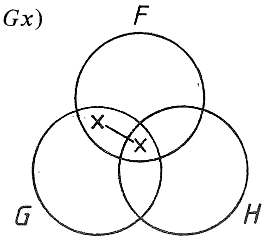
We now set out the representations of **A** and **I** forms in the eight area diagrams.

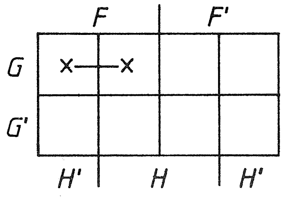
$(\forall x)(Fx \supset Gx)$



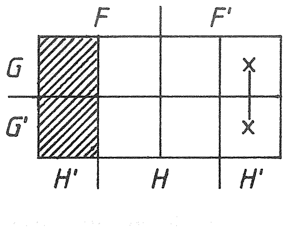


$(\exists x)(Fx\ \&\ Gx)$





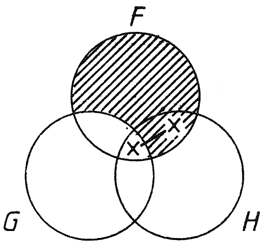
The reverse procedure is also important. Given a diagram which has been filled in we should be able to *read off* the information it contains, e.g. from

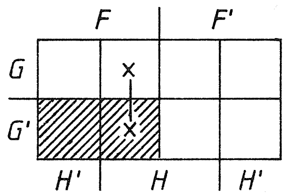


we can read off:

$(\forall x)(Fx \supset Hx)$	*Every F is H*
$(\exists x)(\sim Fx\ \&\ \sim Hx)$	*Some non-F is non-H*

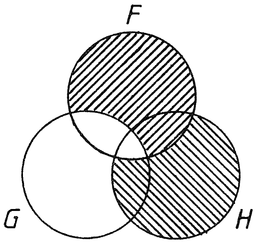
As the above example shows, we can represent several forms conjointly in a diagram. Here is another example. We can represent $(\forall x)(Fx \supset Gx)$ and $(\exists x)(Hx\ \&\ Fx)$ in the one diagram:

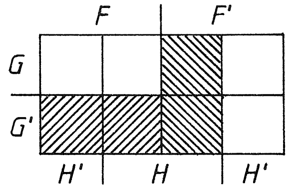




Since the shading covers one end of the disjunction bar it follows that the item which is both H and F must be at the other end. Let's look at another example.

$(\forall x)(Fx \supset Gx)$ and $(\forall x)(Hx \supset Fx)$



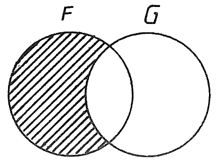


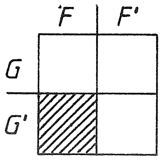
Note that the shading lines for the two formulae have been drawn in different directions.

Inconsistency of a set of forms can be detected by a diagram. If the diagram is filled in for each form in turn and it is not possible to make all the forms in the set true, then the set in inconsistent. For example, consider the set

$\{ (\forall x)(Fx \supset Gx), (\exists x)(Fx \mathbin{\&} \sim Gx) \}$

We begin with the universal $(\forall x)(Fx \supset Gx)$





Now, when we turn to $(\exists x)(Fx \mathbin{\&} \sim Gx)$ we find we cannot fill it in, because we need to place a cross in the shaded area to signify at least one item there. So, the set is inconsistent.

So far we have not considered any formulae in which individual constants, propositional letters or quantifiers have occurred inside the scope of any quantifier. Indeed, we have not looked at any formulae containing propositional letters. The following formulae have one or other of these features.

$p \vee q$, $(\forall x)\, Fx \supset p$, $(\forall x)(Fx \supset p)$, $(\exists x)(Fx \vee Fa)$, $(\forall x)(\exists y)(Fx \mathbin{\&} Gy)$

Such formulae cannot be dealt with using the methods set out above. We will restrict ourselves to formulae in MSF which contain no propositional letters nor free occurrences of individual variables. Even so, we further restrict ourselves to formulae containing no more than one dyadic operator outside the scope of a quantifier. Such formulae we will call *Simple MQL wffs*. The following are non-simple MQL wffs.

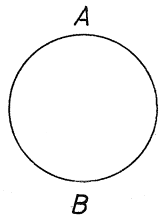
$((\forall x)\, Fx \vee (\exists x)\, Gx) \mathbin{\&} Fa$, $p \supset (\forall x)\, Fx$, Fx, $(\exists x)(Fx \supset Gy)$

We later introduce a diagrammatic method for dealing with non-Simple MQL wffs.

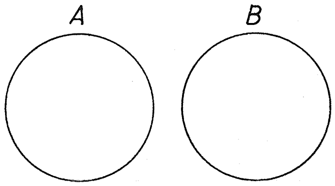
Categorical Diagrams:

As we have seen, Venn circles are always drawn overlapping, and the relationship to be displayed is indicated by filling in certain areas with appropriate markings, Euler diagrams also use circles, but these are not filled in. In contrast, the relationship to be displayed is indicated by the *spatial arrangement* of the circles. Moreover, Euler adopted a *comprehensive existential viewpoint* for his diagrams, assuming that some items exist in all the areas (including the area outside the circles). When two terms are involved there are five different relationships that can be displayed on a single Euler diagram:

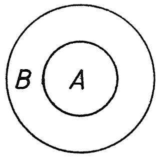
(i)



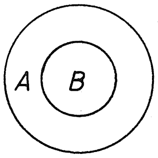
(ii)



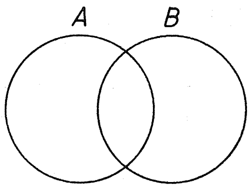
(iii)



(iv)



(v)



(i) Every A is B and every B is A
(ii) No A is B
(iii) All A is B and some B is not A
(iv) All B is A and some A is not B
(v) Some A is not B, some A is B, and some B is not A

According to the comprehensive existential viewpoint it will also be true for each of the above that some items are A, some are B, and some are neither. Euler diagrams are more useful however if we omit this extra requirement, and instead treat the readings above as complete (for instance, with diagram (iii) we will not assume that some items are A).

Adopting this approach, and treating A and B as *sets* (review §9.4 if necessary), the above diagrams can be used to display the following set relations:

(i)	$A = B$	A is identical to B
(ii)	$A \cap B = \{\ \}$	A and B are disjoint
(iii)	$A \subset B$	A is a proper subset of B
(iv)	$B \subset A$	B is a proper subset of A
(v)	$A - B \neq \{\ \} .\&.\ A \cap B \neq \{\ \} .\&.\ B - A \neq \{\}$	A and B properly overlap

Note that we cannot represent the general relationship $A \subseteq B$ (i.e. A is a subset of B) on a single Euler diagram since this relation includes the possibility that $A = B$. However, both $A \subseteq B$ and *Every A is B* can be displayed as a disjunction of the diagrams (i) and (iii): this can be done by placing a "V" between the two figures.

Euler diagrams can be quite useful for finding *counterexamples* to various arguments involving quantifiers and to various purported set theory identities. They are also useful for explaining a limited number of relations in a simple way. In some mathematics texts they are confused with Venn diagrams. Venn diagrams are much more efficient at establishing validity of argument-forms or verifying set theory identities since only a single diagram is required. Recall that when Venn diagrams are used to test set theory identities, shading displays the set under consideration rather than indicating an empty region: in this text we have used dots and lines to distinguish between these different uses of shading on Venn diagrams.

EXERCISE 13.2

1. Use either Karnaugh maps or Venn Diagrams to represent the following forms.

(a) $(\forall x) \sim Fx$
(b) $(\forall x)(\sim Fx \supset \sim Gx)$
(c) $(\exists x)((Gx \;\&\; Fx) \;\&\; Hx)$
(d) $(\exists x)(Fx \vee Gx)$
(e) $(\forall x)(Fx \vee Gx)$
(f) $(\exists x)(Fx \;\&\; (Gx \vee Hx))$
(g) $(\exists x)(Fx \vee (Gx \;\&\; Hx))$
(h) $(\forall x)((Fx \;\&\; Gx) \supset Hx)$
(i) $(\forall x)(Fx \supset (Gx \;\&\; Hx))$
(j) $(\forall x)(Fx \supset \sim(Gx \vee Hx))$
(k) $Fa \;\&\; Gb$
(l) $Fa \vee (\forall x)\, Fx$
(m) $(\exists x)\, Fx \vee (\forall x)\, Fx$
(n) $(\exists x)\, Fx \;\&\; (\forall x)\, Fx$
(o) $(\exists x)\, Fx \;\&\; \sim(\forall x)\, Fx$

2. Use either Karnaugh maps or Venn Diagrams to represent the following sets. If any set is inconsistent state that it is.

(a) $\{(\forall x) \sim Fx, (\forall x)\, Gx\}$
(b) $\{(\forall x)(Fx \supset Gx), (\forall x)(Fx \supset \sim Gx)\}$
(c) $\{(\exists x)(Fx \;\&\; Gx), (\forall x)(Fx \supset \sim Gx)\}$
(d) $\{Fa, Gb, Ha\}$
(e) $\{Fa \vee Gb\}$
(f) $\{\sim Fa, \sim Gb\}$
(g) $\{(\forall x)(Fx \supset Gx), (\forall x)(Gx \supset Hx), (\exists x)(Fx \;\&\; \sim Hx)\}$

3. (a) Draw Venn diagrams to depict the five relations discussed for Euler diagrams.
(b) Use Euler diagrams to represent the form *Every **A** is **B** and every **B** is **C***,
  (i) assuming that *Some **B** is not **A** and some **C** is not **B***.
  (ii) without either of these assumptions.

13.3 VALIDITY OF ARGUMENTS

Diagrams can be used to assess arguments for validity. We will look, in this section, at argument forms in which all the formulae are Simple MQL formulae. The procedure for testing is as follows:

1) A diagram is drawn to cope with all the predicate letters in the argument form.
2) Each premise is filled in on the diagram.
3) The question is then asked: Does the diagram either
  (a) include the conclusion, or
  (b) show an inconsistent set of premises

If the answer to either 3a or 3b is "yes" then the argument is valid; if not then it's invalid.

Example 1:

$$\begin{array}{ll} & (\forall x)(Fx \supset Gx) \\ & (\forall x)(Gx \supset Hx) \\ \hline \therefore & (\forall x)(Fx \supset Hx) \end{array}$$

Step 1. Draw an eight area diagram, (Fig. 1).

Fig. 1

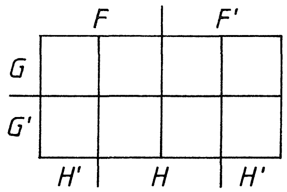
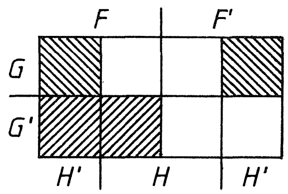


Fig. 2



Step 2. Fill in both premises, to get Fig. 2.

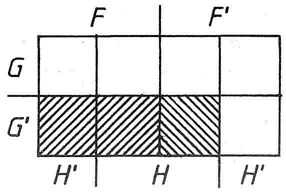
Step 3. The answer to 3a is "yes", so the argument is valid.

Example 2:

$(\forall x)(Fx \supset Gx)$
$(\forall x)(Hx \supset Gx)$
$\therefore\ (\forall x)(Fx \supset Hx)$

Steps 1 and 2: Draw an eight area diagram and fill it in to get Fig. 3.

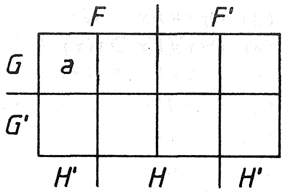
Fig. 3



Step 3. The answer to 3a is "no" and to 3b is "no". For the diagram to represent the conclusion the top leftmost area, area 1, would have to be shaded out. It is not. Since it is not it is easy to construct a counterexample. Some item, say *a*, in that area would leave the premises true but make the conclusion false. So the counterexample is a one item world set out as

	G	F	H
a	1	1	0

or



So the argument is invalid.

Example 3:

$(\exists x)(Fx \ \&\ Gx)$
$(\exists x)(Gx \ \&\ Hx)$
$\therefore\ (\exists x)(Fx \ \&\ Hx)$

Steps 1 and 2: Draw an eight area map and fill it in to get Fig. 4.

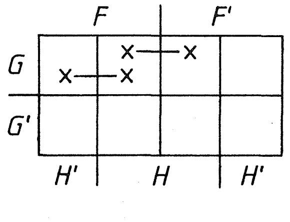
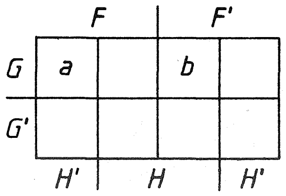


Fig. 4

Step 3: The answers to both 3a and 3b are "no". For us to be able to read off the conclusion we would need either a definite cross in area 2, or one in area 6, or a disjunction bar across from area 2 to 6. The two bars in Fig 4 leave open the possibility that there is an item in area 1, one in area 3, and none in areas 2 or 6.

So, we construct the counterexample. It will be a two-item world as follows

	F	G	H
a	1	1	0
b	0	1	1



EXERCISE 13.3

1. Use Karnaugh Maps or Venn Diagrams to test the following argument-forms for validity. If invalid set out a counterexample.

   (a) $(\forall x)(Hx \supset \sim Fx)$, $(\forall x)(Gx \supset Fx)$ / ∴ $(\forall x)(Hx \supset \sim Gx)$
   (b) $(\exists x)Fx$, $(\exists x)Gx$ / ∴ $(\exists x)(Fx \,\&\, Gx)$
   (c) $(\forall x)(Fx \supset Gx)$ / ∴ $(\forall x)(\sim Gx \supset \sim Fx)$
   (d) $(\forall x)(Fx \supset Gx)$ / ∴ $(\forall x)((Fx \,\&\, Hx) \supset Gx)$
   (e) $(\forall x)(Fx \supset Gx)$ / ∴ $(\forall x)(Fx \supset (Gx \,\&\, Hx))$
   (f) $(\exists x)(Fx \,\&\, Gx)$ / ∴ $(\exists x)Fx$
   (g) $(\exists x)Fx$, $(\forall x)(Fx \supset Gx)$ / ∴ $(\exists x)(Gx \,\&\, Fx)$
   (h) $(\exists x)(Fx \,\&\, Gx)$, $(\forall x)(Gx \supset Hx)$ / ∴ $(\exists x)(Fx \,\&\, Hx)$
   (i) $(\exists x)(Fx \,\&\, \sim Gx)$ / ∴ $(\exists x)(Fx \,\&\, \sim(Gx \,\&\, Hx))$
   (j) $(\exists x)(Hx \,\&\, Fx)$, $(\forall x)(Fx \supset Gx)$ / ∴ $(\exists x)(Hx \,\&\, Gx)$
   (k) $(\forall x)(Fx \supset \sim Gx)$, $(\exists x)(Hx \,\&\, \sim Gx)$ / ∴ $(\exists x)(Hx \,\&\, \sim Fx)$
   (l) $(\exists x)\, Fx$, $(\forall x)(Gx \supset Hx)$, $(\forall x)(Fx \supset \sim Hx)$ / ∴ $(\exists x)(Fx \,\&\, \sim Gx)$
   (m) $(\exists x)\, Fx$, $(\exists x)\, Gx$ / ∴ $(\exists x)(Fx \vee Gx)$
   (n) $(\forall x)(Fx \supset Gx)$, $(\forall x)\, Fx$ / ∴ $(\forall x)\, Gx$
   (o) $(\exists x)(Fx \,\&\, Gx)$ / ∴ $(\exists x)((Fx \vee Hx) \,\&\, (Gx \vee Jx))$

2. Translate the following arguments into MQL, using the dictionary provided, and test for validity by using either Venn Diagrams or Karnaugh Maps.

   (a) Every bag of wheat is counted. Some unmarked bags are bags of wheat. So some of what we counted is unmarked. ($Wx =$ x is a bag of wheat; $Cx = x$ is counted; $Ux =$ x is an unmarked bag.)

(b) Not all price rises can be justified at the moment. Why? Because increases in service charges by Government are price rises, and no such increase can be justified at the moment. ($Px = x$ is a price rise; $Jx = x$ can be justified at the moment; $Ix = x$ is an increase in service charges by Government.)

(c) People said to be successful in our society are well off, and since no-one who receives the pension is well off, it follows that people said to be successful in our society never receive the pension. ($Sx = x$ is a person said to be successful; $Wx = x$ is well-off; $Rx = x$ receives the pension.)

(d) Everyone is imperfect, and everyone is mortal, so some mortals are imperfect. ($Px = x$ is a person; $Ix = x$ is imperfect; $Mx = x$ is mortal.)

(e) No analytic truth is a synthetic truth. Some synthetic truths are *a priori* truths. Hence some *a priori* truths are not analytic. ($Ax = x$ is an analytic truth; $Sx = x$ is a synthetic truth; $Px = x$ is an *a priori* truth)

(f) Some bugs are not insects since insects have six legs but some bugs do not have six legs. ($Bx = x$ is a bug; $Ix = x$ is an insect; $Sx = x$ has six legs.)

(g) Some children who eat acidic foods are hyperactive, and all children who eat foods to which they are allergic are hyperactive. So, there are some children eating acidic foods who are eating foods to which they are allergic. (*Universe* = children; $Ax = x$ eats acidic food; $Hx = x$ is hyperactive; $Fx = x$ eats food to which x is allergic.)

(h) Scientific theories are all contingently true, because scientific theories are empirical propositions, and empirical propositions are all contingently true. ($Tx = x$ is a scientific theory; $Cx = x$ is contingently true; $Ex = x$ is an empirical proposition.)

(i) No one who agitates for reform is apathetic, and everyone who really despises injustice agitates for reform. So, no one who really despises injustice is apathetic. (*Universe* = persons; $Rx = x$ agitates for reform; $Ax = x$ is apathetic; $Dx = x$ despises injustice).

(j) Since no one who is looking for security will expect a windfall profit, it follows that some who expect a windfall profit will not invest at low interest rates, because everyone who invests at low interest rates is looking for security. (*Universe* = persons; $Lx = x$ is looking for security; $Ex = x$ expects a windfall profit; $Ix = x$ invests at low interest rates.)

(k) Some river-bank land is not to be built on. This is because all river-bank land is flood prone and no flood prone land is to be built on. ($Rx = x$ is riverbank land; $Bx = x$ is to be built on; $Fx = x$ is flood prone land.)

(l) All of the bantam's eggs are fertile, and eggs which hatch are fertile eggs. So some of the bantam's eggs will hatch. ($Bx = x$ is one of the bantam's eggs; $Fx = x$ is a fertile egg; $Hx = x$ will hatch.)

(m) Some items on student records are kept secret. Since no items on student records are released without special permission, it follows that some items kept secret are not released without special permission. (*Universe* = items on student records; $Sx = x$ is kept secret; $Rx = x$ is released without special permission).

(n) All social welfare programs are stop-gap measures, because stop-gap measures are all measures brought in during an emergency to help people, and all measures brought in during an emergency to help people are social welfare programs. ($Px = x$ is a social welfare program; $Sx = x$ is a stop-gap measure; $Bx = x$ is brought in during an emergency to help people.)

(o) Everyone trying to work will find satisfaction. No one who sits by and watches will find satisfaction. Hence, some who sit by and watch are not trying to work. (*Universe* = persons; $Tx = x$ is trying to work; $Sx = x$ finds satisfaction; $Wx = x$ sits by and watches.)

(p) No one who is alert to public opinion will introduce unpopular changes in education just before an election. Some who are alert to public opinion do not win office. So, no one who introduces unpopular changes in education just before an election wins office. (*Universe* = persons; $Ax = x$ is alert to public opinion; $Ix = x$ introduces unpopular changes in education just before an election; $Wx = x$ wins office.)

(q) No analytic truth is a synthetic truth. All analytic and synthetic truths are propositions. All propositions which are true by virtue of the meanings of terms are analytic truths. So, synthetic truths are propositions not true by virtue of the meanings of terms. ($Ax = x$ is an analytic truth; $Sx = x$ is a synthetic truth; $Px = x$ is a proposition; $Tx = x$ is true by virtue of the meanings of terms.)

(r) No efficient bureaucrat fusses about. The red Queen's ministers are all rabbits. Rabbits fuss about. So none of the red Queen's ministers are efficient bureaucrats who are punctual. ($Bx = x$ is an efficient bureaucrat; $Fx = x$ fusses about; $Mx = x$ is a minister of the red Queen; $Rx = x$ is a rabbit; $Px = x$ is punctual.)

(s) No EIO syllogism is invalid. Since there are figure 1 and figure 2 syllogisms which are invalid, it follows that there are figure 1 and figure 2 syllogisms which are not EIO syllogisms. ($Ex = x$ is an EIO syllogism; $Ix = x$ is invalid; $Ox = x$ is a figure 1 syllogism; $Tx = x$ is a figure 2 syllogism.)

(t) No-one who has paid tax and applies for the dole should have problems getting money. Some people who apply for the dole do have problems getting money even though they have paid tax. So it follows that some people do have problems getting money even though they should not have such problems. ($Px = x$ is a person; $Tx = x$ has paid tax; $Sx = x$ should have problems getting money; $Ax = x$ applies for the dole; $Hx = x$ does have problems getting money.)

13.4 MORE COMPLEX CASES IN MQT

Consider the three formulae, (1) to (3)

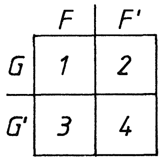
$$(\forall x)\, Fx \lor (\forall x)\, Gx \qquad (1)$$
$$p \supset (\exists x)\, Gx \qquad (2)$$
$$p \lor q \qquad (3)$$

We now introduce the *indirect method* of representing these formulae. We will not represent these formulae in diagrams, but will *use* diagrams to represent these formulae in PL.

First we set out a fixed dictionary, and use the numerals 1, 2, 3 ... as fixed propositional constants:

1 = *area* 1 *is null*
2 = *area* 2 *is null*
3 = *area* 3 *is null*
⋮
n = *area* n *is null*

Consider Diagram 1 with its areas numbered:



Diag. 1.

By using the dictionary and the usual shading for $(\forall x)\ Fx$ we can represent $(\forall x)\ Fx$ *with respect to Diagram 1* as (4).

$$2 \ \&\ 4 \tag{4}$$

(4) reads as (5)

Area 1 and area 4 are null (5)

Similarly $(\forall x)\ Gx$ is represented by (6) (with respect to Diagram 1).

$$3 \ \&\ 4 \tag{6}$$

So (1) is represented by (7) (with respect to Diagram 1)

$$(2 \ \&\ 4) \lor (3 \ \&\ 4) \tag{7}$$

In the same diagram $(\exists x)\ Gx$ is represented by (8)

$$\sim 1 \lor \sim 2 \tag{8}$$

which reads as (9)

Either area 1 is not empty or area 2 is not empty (9)

So we represent (2) as (10)

$$p \supset (\sim 1 \lor \sim 2) \tag{10}$$

Note that propositional variables represent themselves. So (3) represents (3) in Diagram 1, and in *every* diagram.

We use these PL representations to test formulae for MQT-Necessity. Consider (11)

$$(\forall x)(Fx \supset Gx) \supset ((\forall x)\ Fx \supset (\forall x)\ Gx) \tag{11}$$

In Diagram 1 we represent (11) by (12)

$$3 \supset ((2 \ \&\ 4) \supset (3 \ \&\ 4)) \tag{12}$$

This is a tautology. So we can conclude that (11) is an MQT-Necessity. Consider also (13)

$$(\exists x)(Fx \ \&\ Gx) \supset ((\exists x)\ Fx \ \&\ (\exists x)\ Gx) \tag{13}$$

This is represented in Diagram 1 by (14)

$$\sim 1 \supset ((\sim 1 \lor \sim 3) \ \&\ (\sim 1 \lor \sim 2)) \tag{14}$$

Once again, this is a tautology, so (13) is an MQT-Necessity.

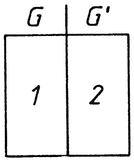
Unfortunately, there is one complication. In our representations of (11) and (13) we have not had to worry about whether the domain is empty or non-empty. But formula (15) is an MQT-Necessity for just the reason that, in MQT, we operate in non-empty domains.

$$(\forall x)\ Gx \supset (\exists x)\ Gx \tag{15}$$

In Diagram 1 we represent (15) by (16)

$$(3 \,\&\, 4) \supset (\sim 1 \lor \sim 2) \tag{16}$$

(16) does not have the form of a tautology. The counterexample is the top row of the truth-table: $1 = 1, 2 = 1, 3 = 1, 4 = 1$. That row represents the domain's being empty. We can overcome this problem by always using a truth-table with no top row. We use a *non-null truth table*. If we use MAV or truth trees, then any model or path in which every numeral of the formula (or argument) is true is closed. So (15) is an MQT-Necessity. Of course, we need test formulae only in the smallest diagram in which they can be represented. So we could test (15) in Diagram 2.



Diag. 2

For (15) we test (17)

$$2 \supset \sim 1 \tag{17}$$

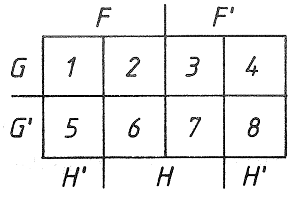
which is always true in a non-null truth-table.

This method of testing can easily be extended to arguments.

We test

$$\begin{array}{rl} & (\forall x)(Fx \supset Gx) \\ & (\forall x)(Gx \supset Hx) \\ \hline \therefore & (\forall x)(Fx \supset Hx) \end{array}$$

in Diagram 3



Diag. 3

by testing:

$$\begin{array}{rl} & 5 \,\&\, 6 \\ & 1 \,\&\, 4 \\ \hline \therefore & 1 \,\&\, 5 \end{array}$$

Care must be taken when representing formulae such as *Fa* in Diagram 3. In some arguments *Fa* might represent itself. In others we need $(\sim 1 \lor \sim 2 \lor \sim 5 \lor \sim 6)$ for *Fa.*

Despite the effectiveness of this testing method we still cannot use it generally for MQT. We cannot represent formulae like (18), (19) and (20)

$$(\forall x)(Fx \supset p) \tag{18}$$
$$(\forall x)(Fx \supset Ga) \tag{19}$$
$$(\forall x)(Fx \supset (\forall y)\, Gy) \tag{20}$$

These are the formulae in which propositional letters, individual constants, or quantifiers occur within the scope of a quantification.

The only way to deal with such is to have a *minimum scope procedure*. That is, we need a way of producing formulae which are logically equivalent to (18), (19) and (20) and their ilk, but in which no propositional letter, individual constant, or quantifier is within the scope of a quantification. There is such a procedure, as we have seen in § 12.4. It would produce, for example (21) to (23) from (18) to (20) respectively:

$$(\exists x)\, Fx \supset p \tag{21}$$
$$(\exists x)\, Fx \supset Ga \tag{22}$$
$$(\exists x)\, Fx \supset (\forall y)\, Gy \tag{23}$$

NOTES

The *indirect method* of representing MQL wffs is due to Phillip Staines.

EXERCISE 13.4

1. Translate the following arguments into MQL, using the dictionary provided, and test them for validity by using the indirect method of diagrammatic representation.
   - (a) Names are not revealed to the press because all names are kept secret to protect the victims and whatever is kept secret to protect victims is never revealed to the press. ($Nx =$ x is a name; $Rx = x$ is revealed to the press; $Px = x$ is kept secret to protect the victims.)
   - (b) Whoever has lost a game will go to the second round with fewer points. Whoever goes to the second round with fewer points will play under a penalty. So some who play under a penalty have lost a game. ($Lx = x$ has lost a game; $Gx = x$ will go to the second round with fewer points; $Px = x$ will play under a penalty.)
   - (c) Debaters are never unbiased and advocates are debaters. So some unbiased men are not advocates. ($Dx = x$ is a debater; $Ux = x$ is unbiased; $Ax = x$ is an advocate.)
   - (d) Some reporters are not likely to be fair to members of groups of eccentrics. Reporters are looking for interesting stories. So some who look for interesting stories are not likely to be fair to members of groups of eccentrics. ($Rx = x$ is a reporter; $Fx = x$ is likely to be fair to members of groups of eccentrics; $Lx = x$ is looking for interesting stories.)
   - (e) Some soldiers are heroes, but some soldiers are not brave. So, some heroes are not brave. ($Sx = x$ is a soldier; $Hx = x$ is a hero; $Bx = x$ is brave.)
   - (f) All members are both officers and gentlemen. All officers are fighters. Only a pacifist is either a gentleman or not a fighter. No pacifists are gentlemen if they are fighters. Some members are fighters if and only if they are officers. Therefore not all members are fighters. ($Mx = x$ is a member; $Ox = x$ is an officer; $Gx = x$ is a gentleman; $Fx = x$ is a fighter; $Px = x$ is a pacifist.)

(g) If the University's rules are equitable, then students who have either academic qualifications or demonstrable ability will be eligible for entry. Since students who have academic qualifications or have demonstrable ability are not all admitted, it follows that even if the University's rules are equitable, not every student who is eligible for entry will be admitted. (*Universe* = Students; R = The University's rules are equitable; $Qx = x$ has academic qualifications; $Dx = x$ has demonstrable ability; $Ex = x$ is eligible for entry; $Ax = x$ is admitted.)

(h) All applicants who are women will be disappointed if they are not hired. Some women will be hired, but all applicants are women, so it follows that some applicants will be disappointed. ($Ax = x$ is an applicant; $Wx = x$ is a woman; $Dx = x$ is disappointed; $Hx = x$ is hired.)

(i) If there are innovations in education and some teachers are not re-trained to cope, then some students do not benefit. Only those who benefit are seen as evidence in favour of innovation. Since there are innovations in education and some students are not seen as evidence in favour of innovation, it follows that teachers are not all re-trained to cope. (I = There are innovations in education; $Tx = x$ is a teacher; $Rx = x$ is re-trained to cope; $Sx = x$ is a student; $Bx = x$ benefits; $Fx = x$ is seen as evidence in favour of innovation.)

(j) Either all empiricists are illogical or no philosophers are realistic. Thus, if some philosophers are empiricists, then some empiricists are either illogical or unrealistic. ($Ex = x$ is an empiricist; $Ix = x$ is illogical; $Px = x$ is a philosopher; $Rx = x$ is realistic.)

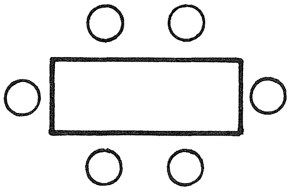
(k) No dentists are quantity surveyors, and no garbage pickers are dentists. So it follows that no garbage pickers are quantity surveyors. ($Gx = x$ is a garbage picker; $Dx = x$ is a dentist; $Qx = x$ is a quantity surveyor.)

(l) Only beliefs involving God are, on Brown's definition, religious beliefs. But since some eastern religious beliefs do not involve God, it follows that some eastern religious beliefs are not, on Brown's definition, religious beliefs. ($Gx = x$ is a belief involving God; $Bx = x$ is, on Brown's definition, a religious belief; $Ex = x$ is an eastern religious belief.)

(m) Not all scientists are convinced that nuclear power is safe. This is so, because some opponents of nuclear power are scientists, and no one who is convinced that nuclear power is safe is an opponent of nuclear power. ($Sx = x$ is a scientist; $Cx = x$ is convinced that nuclear power is safe; $Ox = x$ is an opponent of nuclear power.)

(n) Every detail of the new system is in this book. Since no detail of the new system has been put on file, it follows that some things in this book have not been put on file. ($Dx = x$ is a detail of the new system; $Ix = x$ is in this book; $Px = x$ has been put on file.)

Puzzle 13 Three couples go to dinner. They sit around a rectangular table, one couple on each side, and the third couple sit opposite each other at the ends of the table. The three men are Fred, George and Henry, and their partners are, in alphabetical order, Anne, Beth and Cath. From the following work out who is with whom, and who is sitting at the ends of the table:

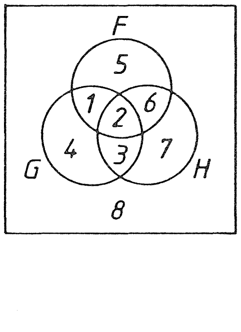


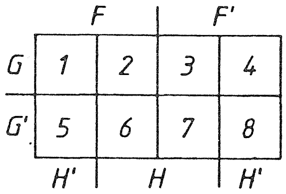
1. Fred is immediately to the left of Anne.
2. Henry is opposite Beth.
3. George is not opposite Fred.
4. Beth is neither to the immediate left nor to the immediate right of George.

13.5 SUMMARY

Various *logic diagrams* may be used to represent and test formulae in MQT and set theory. Some of these are of the *fill-in* type (e.g., Venn diagrams and Veitch-Karnaugh maps) while others are *categorical* (e.g., Euler diagrams). With the fill-in variety, the background diagram depends only on the predicates or sets involved; the relationship between these as described by the formula is then displayed by adding marks to this background. For *Venn diagrams* and *Karnaugh maps* these marks may be of five kinds: *individual constants* are placed only in areas predicated of them; *shading* or a *small circle* indicates the area is empty; a *cross* indicates the area contains at least one item; and a *disjunction bar* joins alternatives.

A Venn or Karnaugh diagram for n predicates or sets will have 2^n distinct areas: on a Karnaugh map these are numbered in usual reading order (left to right, downwards); Venn areas are numbered to correspond with these. Venn diagrams consist of overlapping circles or ellipses for each predicate or set, with a complementary rectangular outline (sometimes omitted). Karnaugh maps have a rectangular array of cells whose coordinates are the predicates (or sets) or their complements. The example below shows first a Venn then a Karnaugh diagram for F, G, H with the areas numbered.





Information can be both *entered on* and *read off* Venn and Karnaugh diagrams in accordance with the five conventions mentioned earlier: a clash with these conventions when coding a set of formulae onto one diagram indicates the set is *inconsistent*.

MQL wffs in MSF which contain no propositional letters, no free IVs, and at most one dyadic operator outside the scope of a quantifier are called *Simple MQL wffs*. These may be tested on a single Venn or Karnaugh diagram.

Unlike Venn diagrams, *Euler diagrams* display relationships by the *spatial arrangement* of circles instead of filling in areas with marks. In addition, items are assumed to exist in certain areas. When two predicates or sets are involved, five different relationships can be displayed on a single Euler diagram (see §13.2). Further relations can be displayed by disjoining diagrams. Though useful for finding counterexamples and explaining simple relations, Euler diagrams are not very useful for establishing validity or set-theory identities.

Venn and Karnaugh diagrams may be used to *test for validity* of Simple MQL argument-forms by filling in all the premises on the one diagram: the argument-form is valid iff either the conclusion may be read off from the result or the premises were inconsistent. This method may be extended to test arguments, with the additional requirement that the countermodel must be possible.

MQL wffs which are not Simple but are in MSF may be treated by the *indirect diagrammatic method*. Here the appropriate Venn or Karnaugh diagram is selected, with areas numbered 1, ..., n as usual. The formula is then translated into PL with respect to this diagram by using the numerals *1*, ..., n as propositional constants, where *1* = *area* 1 *is empty*, etc. Testing may now be done by PC methods (e.g., MAV or tables) except that we assume not all areas are empty (so the model where each of *1*, ..., n = 1 is crossed off and cannot be used for a counterexample).

14 Quantification Theory

14.1 RELATIONS AND MULTIPLE QUANTIFICATION

The quantifier theory which we have been studying is not able to cope with propositions such as (1), (2) and (3).

Somebody loves somebody. (1)
Every event has some cause. (2)
For any two people, if the first is a cousin of the second then the second is a cousin of the first. (3)

In order to deal with these, and others like them, we have to take account of *relations*. In each of the propositions, (4) to (10), it is asserted that a relationship of some kind holds between two items.

Anne loves Bill. (4)
Charles is a brother of Sue. (5)
Ken and Judy are cousins. (6)
David is an ancestor of Paul. (7)
Australia is smaller than Canada. (8)
Cairns is west of Canberra. (9)
The passage is in the book. (10)

These relations are *dyadic* relations because they hold between *two* items. Some relations are triadic, because they hold between three items. For example:

John is between Bill and Fred. (11)
Sue promised to give this pearl to Sally (12)

Some relations are tetradic, because they hold between four items. Consider (13).

Toni will share the prize with Michelle and Sandra. (13)

In general we refer to *n-adic* relations where the relation holds between *n* items; *n* then gives us the *adinity* of the relation. *n-adic* relations are also called *n-place predicates*. This enables us to talk about one-place, two-place, three-place etc. predicates.

Our main interest and focus of attention will be on *dyadic predicates*. Many of the dyadic predicates we meet in ordinary conversation can be expressed in either of two forms. The two forms are called the *active voice* and the *passive voice*. For example, in (14) and (15) we have the two ways of expressing the same propositions

"Anne loves Bill." (14)
"Bill is loved by Anne." (15)

(14) is in the *active* voice, but (15) is in the *passive.* The word "by" often occurs when the passive voice is used. In the following pairs the first is in the active voice and the second in passive.

Ms Jones owns the flats. (16a)
The flats are owned by Ms Jones. (16b)

Harry threw the ball. (17a)
The ball was thrown by Harry. (17b)

Many dyadic predicates do not have such alternative forms of expression in English. For our purposes we will consider them to be in the active voice. For example:

John is smaller than Sue. (18)
Alan is a brother of Bill. (19)
Coonabarabran is west of Goondiwindi. (20)

Many propositions involving relations are expressed in one of the **A**, **E**, **I** or **O** forms. Consider the following, (21) to (24):

Every logician admires Russell. (21)
No logician admires Russell. (22)
Some logician admires Russell. (23)
Some logician does not admire Russell. (24)

They are **A**, **E**, **I** and **O** respectively. Since the relation is expressed in the active voice in each of (21) to (24) we will say that they are, respectively, **A** *active,* **E** *active,* **I** *active* and **O** *active*. The following, (25) to (28), are **A** *passive*, **E** *passive*, **I** *passive* and **O** *passive* respectively.

Every logician is admired by Russell. (25)
No logician is admired by Russell. (26)
Some logician is admired by Russell. (27)
Some logician is not admired by Russell. (28)

Consider the following four propositions, (29 to 32):

Every logician admires every genius. (29)
Every logician admires no genius. (30)
Every logician admires some genius. (31)
Every logician admires not all geniuses. (32)

Each of the above four is **A** *active*, but each has a quantifier in its *consequent*. We can describe (29) as an **A** *over* **A** *active* proposition, (30) as an **A** *over* **E** *active*, (31) as an **A** *over* **I** *active*, and (32) as an **A** *over* **O** *active*. Note that the negated **A** is treated as **O**. Look carefully at each of (33) to (36) and see if you can decide how each is to be described before looking at the answers below. In this notation the first letter and the last word give the *overall* form.

No logician is admired by every genius. (33)
No logician is admired by no genius. (34)
Some logician is admired by some genius. (35)
Some logician is admired by not all geniuses. (36)

(33) is **E** *over* **A** *passive*, (34) is **E** *over* **E** *passive*, (35) is **I** *over* **I** *passive*, (36) is **I** *over* **O** *passive.*

Consider each of the following four **A** *over* **A** *active* propositions, (37) to (40):

Every logician who respects everyone admires every writer. (37)
Every logician who respects no one admires every writer. (38)
Every logician who respects some one admires every writer. (39)
Every logician who respects not everyone admires every writer. (40)

Each of these has a quantifier in its *antecedent*. We can describe

(37) as an **A** *by* **A** *active over* **A** *active*,
(38) as an **A** *by* **E** *active over* **A** *active*,
(39) as an **A** *by* **I** *active over* **A** *active*, and
(40) as an **A** *by* **O** *active over* **A** *active.*

We are using "by" as an abbreviation for "qualified by" to indicate the form of the antecedent of a quantified conditional, and "over" as an abbreviation for "has scope over" to indicate the form of the consequent. Where it is a quantified conjunction then "by" indicates the form of the left conjunct, and "over" the form of the right conjunct.

We now set out four further examples, (41) to (44). Try to describe them before looking at the answers below.

Some writer admires all geniuses. (41)
Some writer who likes all authors admires all geniuses. (42)
No writer who likes all authors is admired by all geniuses. (43)
Every author who likes some writer admires some geniuses. (44)

(41) is **I** *over* **A** *active*, (42) is **I** *by* **A** *active over* **A** *active*, (43) is **E** *by* **A** *active over* **A** *passive*, (44) is **A** *by* **I** *active over* **I** *active.*

We will make considerable use of this descriptive terminology in the section on translation, §14.5.

NOTES

The taxonomy used in this section was created by Corinne Miller.

We have used "adinity" rather than "adicity" just in case some students get to think they are learning about the PH of predicates. Some authors use the latin based terms *unary*, *binary*, *ternary*, etc.

EXERCISE 14.1

1. What is the adinity of the relations in the following propositions?
   (a) Susan is a cousin of Sally.
   (b) Elizabeth is a parent of Charles.
   (c) Fred is the spouse of Wilma.
   (d) Cliff negotiates between Ian and Mal.
   (e) Tony stands to the left of Margaret between Shirley and Karl.

2. Which of the following are in the active voice and which are in the passive? Rewrite all the passive ones in the active.
   (a) Sally respects Sue.
   (b) Mike was struck by the ball.
   (c) Dennis was given out by the umpire.
   (d) Robin is Chris' sister.
   (e) Everyone is respected by someone or other.

3. Describe each of the following in terms of **A E I** and **O**, active and passive.
   - (a) Some politicians take note of Milton.
   - (b) Not all viewers are watching the *Two Ronnies.*
   - (c) Some viewers are being monitored by the rating firm.
   - (d) No problems in logic confront every student.
   - (e) Every student is confronted by some problem.
   - (f) There are some problems which every student solves.
   - (g) Some logician is admired by all logicians.
   - (h) Every logician who is loved by someone solves every problem.
   - (i) No student who solves some problems will ignore every problem.
   - (j) Every student who solves no problems will admire Aristotle.

14.2 SYNTAX FOR QL

We now define the formulae for a language called *Quantificational Language* (QL). This language includes all the formulae of MQL. We can see QL as an *extension* of MQL. In order to extend MQL we need to do two things to the definitions for MQL. First we must alter the primitive symbols by changing the capital letters. Then we need to change clause (B2M). The capital letter list is as follows:

$F^n, G^n, H^n, I^n, J^n, \ldots$	Any number of capital letters with superscripts

We need the following terminology

S^n	denotes any primitive drawn from F^n, G^n, ...
$\mathrm{s}_1 \ldots \mathrm{s}_n$	denotes n primitives (not necessarily different) drawn from either or both of $x, y, \ldots$ and $a, b, \ldots$

Instead of (B2M) we need (B2)

Anything of the form $\mathrm{S}^n\ \mathrm{s}_1\ \ldots\ \mathrm{s}_n$ is a wff. (B2).

Examples of the correct use of (B2) are

$$F^1a,\ F^1x,\ F^2ab,\ F^2ax,\ F^2xa,\ F^2xy,\ F^2xx,\ F^3xay,\ F^4aaxb$$

We can now generate formulae such as:

$$(\exists x)\,F^1x,\ \ (\forall x)(\exists y)\,F^2xy,\ \ (\forall x)(F^1x \supset (\exists y)(F^2ya \,\&\, G^2xy))$$

Here is an example of a QL-wff assembly line:

1. q	B1
2. F^1y	B2
3. G^2yx	B2
4. $(F^1y \,\&\, G^2yx)$	2, 3, R &
5. $(\exists y)(F^1y \,\&\, G^2yx)$	4, R $\exists$
6. H^2xa	B2
7. $(H^2xa \vee q)$	1, 6, R$\vee$
8. $((H^2xa \vee q) \supset (\exists y)(F^1y \,\&\, G^2yx))$	5, 7, R $\supset$
9. $(\forall x)((H^2xa \vee q) \supset (\exists y)(F^1y \,\&\, G^2yx))$	8, R $\forall$

The same principles operate in QL as in MQL for bound and free occurrences of individual variables. Binding can be displayed with lines as follows. To avoid lines crossing, if possible, we use lines over as well as under the formula.

$$(\forall x)(\forall y)\, F^2xy$$

$$(\forall x)((H^2xa \vee q) \supset (\exists y)(F^1y \,\&\, G^2yx))$$

Notice how both quantifiers, in our examples, have binding lines running into the same atomic sub-formulae: G^2yx and F^2xy.

EXERCISE 14.2

1. Set out assembly lines for the following formulae.
   - (a) $(\forall x)(F^1x \supset F^2xa)$
   - (b) $((\exists x)\, F^2xa \supset p)$
   - (c) $(\exists x)(\exists y)\, F^2xy \equiv (\forall x)(G^1a \supset H^1b))$
   - (d) $(\sim(\exists y)(\exists x)(\forall z)\, G^3xyz \supset (\forall x)(\exists z) \sim(F^2\, xz \,\&\, H^2xz)$
   - (e) $(\forall x)((p \supset (\exists x)\, F^1x) \vee \sim G^2xy)$
2. Draw binding lines for the formulae in Q. 1.

14.3 SEMANTICS FOR QT

The symbols of QL are given meaning in the same way as we gave meaning to the symbols of MQL. The resulting system is called *Quantification Theory* (QT).

We need consider only the new set of capital letters and the quantifiers.

The capital letters with the superscript 1 are the same as the predicate letters of MQT. They will be used to stand for properties just as before. Capital letters with superscripts of 2 or greater will be used to stand for relations. We will still refer to the capital letters, no matter what the superscript, as *predicate letters*. Those with the superscript 1 will be picked out by the names *monadic predicate letters* or *one-place predicate letters*. We set out the following table of terminology

$\mathbf{S}^1$	monadic *or* one-place predicate letters
$\mathbf{S}^2$	dyadic *or* two-place predicate letters
$\mathbf{S}^3$	triadic *or* three-place predicate letters
$\mathbf{S}^4$	tetradic *or* four-place predicate letters
$\mathbf{S}^5$	five-place predicate letters
⋮	⋮

The formulae (1) means that a has the relation F to b.

$$F^2ab \tag{1}$$

If we have the following dictionary:

a = Alan
b = Bill
F^2xy = x is friendly to y

then (1) means

Alan is friendly to Bill. (2)

Similarly (3) means (4)

$$F^2ba \tag{3}$$

Bill is friendly to Alan (4)

It is important to note that just as (1) and (3) have a and b reversed, so do (2) and (4). Consider the further dictionary entry

$L^2xy =$ x is larger than y

It is clear here also that the order of x and y is very important. So (5) means (6), and (7) means (8).

$$L^2ab \tag{5}$$

Alan is larger than Bill (6)

$$L^2ba \tag{7}$$

Bill is larger than Alan (8)

Given our familiarity with the English language it is more natural to place the dyadic predicate letters *between* the individual letters. Also, the adinity of a predicate may be obtained by counting the number of associated individual letters, so the superscripts are redundant. So we relax our rules for QL-wffs as follows

Practical Concession: *Dyadic predicate letters may be placed between individual letters:*
Predicate superscripts may be deleted.

So, (5) may be written as either (9) or preferably as (10), and (6) as either (11) or preferably (12).

$$Lab \tag{9}$$

$$aLb \tag{10}$$

$$Lba \tag{11}$$

$$bLa \tag{12}$$

We now set out a dictionary for predicate letters. Wherever possible our dictionary sets dyadic relations in the *active voice.*

Px = x is a person
Hx = x is happy
Jx = x jumps
xLy = x is larger than y
xRy = x respects y

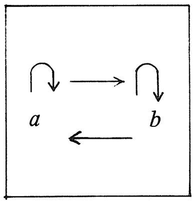
a = Alan
b = Bill
c = Caron

e.g.,

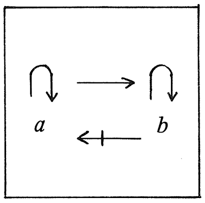
Alan jumps	Ja
Bill is not larger than Caron	$\sim bLc$
Alan respects Bill	aRb

Let us now look at some two-item worlds. On the diagrams we will represent the relation xRy by $x \rightarrow y$ and its negation by $x \nrightarrow y$

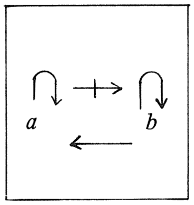
A



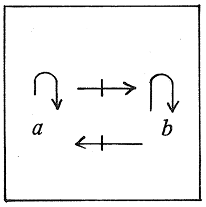
B



C

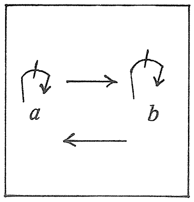


D



In each of A, B, C and D, the individuals respect themselves. That remains constant. But of course that could change. Consider E.

E



Neither *a* nor *b* has self respect.

In worlds A and E *aRb* and *bRa* are true. There is mutual respect. In B *aRb* is true but *bRa* is not. In (c) we have the reverse of B. In D neither respects the other. Each of these worlds can be described by Cayley tables. We set out the following five tables and a key.

A

R	*a*	*b*
a	1	1
b	1	1

B

R	*a*	*b*
a	1	1
b	0	1

C

R	*a*	*b*
a	1	0
b	1	1

D

R	*a*	*b*
a	1	0
b	0	1

E

R	*a*	*b*
a	0	1
b	1	0

KEY

R	*a*	*b*
a	*aRa*	*aRb*
b	*bRa*	*bRb*

Look carefully at each table. The truth values are for the formulae in that position in the key.

In general tables for dyadic predicates are set out as follows:

R	*a*	*b*	*c*	*d* ...
a				
b				
c				
d				
⋮				

Each dyadic predicate has its own table. The items are listed in the *same order* across the top and down the side. The values are inserted according to the key set out below:

R	*a*	*b*	*c*	*d*	...
a	*aRa*	*aRb*	*aRc*	*aRd*	...
b	*bRa*	*bRb*	*bRc*	*bRd*	
c	*cRa*	*cRb*	*cRc*	*cRd*	
d	*dRa*	*dRb*	*dRc*	*dRd*	
⋮	⋮				

For a one item world, where the item is named "*a*", there is only one entry where *aRa* is in the table below

R	*a*
a	*aRa*

Tables for triadic, tetradic, etc., predicates are set out like truth-table matrices with one column of values. For one item worlds there is only one space into which either 1 or 0 is entered. For example:

	F^3
aaa	1

	F^4
aaaa	1

But there are massively more spaces for values in any two-item world. We set out the blank table for F^3 in a two-item world:

	F^3
aaa	
aab	
aba	
abb	
baa	
bab	
bba	
bbb	

For any predicate of *n* places, in any world of *i* items there will be i^n spaces for values. So, for example, a tetradic predicate in a five-item world would have 5^4 or 625 spaces in its table.

We can describe a two-item possible world, using the earlier dictionary, by means of the following three tables (taken conjointly).

(F)

	P	*H*	*J*
a	1	1	0
b	1	0	1

R	*a*	*b*
a	1	0
b	0	1

L	*a*	*b*
a	0	1
b	0	0

Note that there is one table for all the monadic predicates, but *one each* for the dyadic predicates. By referring to our dictionary we see that in this possible world, *F*, there are two people, Alan and Bill. Alan is happy but does not jump. Bill is not happy but does jump. Both Alan and Bill have self-respect, but neither respects the other. It is clear that Alan is the larger of the two.

From this point on, in this section, we will restrict our universe of discourse to people.

Universe = persons

We will also use the notation $[a, ..., b]$ to represent any world where items are $a, ..., b$ i.e. any world whose domain is $\{a, ..., b\}$.

For finite worlds, the quantifications can be eliminated in exactly the same way as for MQT. So, in $[a, b]$ (13) is equivalent to (14)

$$(\forall x)(\forall y)\, xRy \tag{13}$$

$$aRa \,\&\, aRb \,\&\, bRa \,\&\, bRb \tag{14}$$

To show this equivalence we now eliminate the quantifications step by step.

$$(\forall x)(\forall y)\, xRy \tag{13}$$

$$\Leftrightarrow \quad (\forall x)(xRa \,\&\, xRb) \qquad \text{(eliminate } (\forall y))$$

$$\Leftrightarrow \quad aRa \,\&\, aRb \,\&\, bRa \,\&\, bRb \tag{14}$$

Similarly (15) is equivalent to (16) in $[a, b]$

$$(\forall x)(\exists y)\, xRy \tag{15}$$

$$\Leftrightarrow \quad (\forall x)(xRa \vee xRb) \qquad \text{(eliminate } (\exists y))$$

$$\Leftrightarrow \quad (aRa \vee aRb) \,\&\, (bRa \vee bRb) \tag{16}$$

In the world (F), (14) is false, so (13) is false. (14) asserts that, in (F), each person respects everyone (including himself) i.e. (17).

Everyone respects everyone (17)

(17) is false in (F).

But (16) is true in (F). Check it out carefully. (16) says that a respects either a or b and b respects either a or b. In other words, each item respects at least one item or other. We can read (15) as (18)

Everyone respects someone or other (18)

In (F), (18) is true.

Now consider (19) and its equivalent (20).

$$(\exists y)(\forall x)\, xRy \tag{19}$$

$$\Leftrightarrow \quad (\exists y)(aRy \,\&\, bRy) \qquad \text{(eliminate } (\forall x))$$

$$\Leftrightarrow \quad (aRa \,\&\, bRa) \vee (aRb \,\&\, bRb) \tag{20}$$

(20) is false in (F). (20) says that either a and b both respect a or a and b both respect

b. In other words, at least one item is respected by every item. We read (19) as (21).

Someone is respected by everyone. (21)

When there are two quantifications, one universal and the other existential in sequence, then one sequence will be equivalent to a conjunction of disjunctions, the other to a disjunction of conjunctions. Compare (16) and (20). So the order of quantifications, when they are of different kinds, is very important. Consider (22) and its expansion for (F): (23)

$$(\exists x)(\forall y)\, xRy \tag{22}$$

$$\Leftrightarrow (\exists x)(xRa \,\&\, xRb)$$

$$\Leftrightarrow (aRa \,\&\, aRb) \vee (bRa \,\&\, bRb) \tag{23}$$

From (23) we see that (22) is read as (24).

There's at least one person who respects everyone (24)

In (F) (23) is false, so also is (22). Finally we consider (25), its expansion (26), and reading as (27).

$$(\exists x)(\exists y)\, xRy \tag{25}$$

$$aRa \vee aRb \vee bRa \vee bRb \tag{26}$$

Someone respects someone. (27)

(26) is true in (F). So (25) is true in (F).

For MQT, all logical problems with which we would be concerned can be resolved in finite worlds. This is *not* so for QT. So we have to consider worlds with infinitely many items in them. The first obvious thing about infinite worlds is that we cannot describe them by means of tables such as we used for (F). We can provide a *partial description* only. The second obvious thing is that we cannot expand formulae for infinite worlds. But we can set out truth conditions:

Definition: **$(\forall \nu)\, \alpha$ is true iff every itemization of α is true**

$(\exists \nu)\, \alpha$ is true iff at least one itemization of α is true.

It should be noted that therse conditions apply to finite as well as to infinite worlds. It turns out that we need concern ourselves with infinite worlds only when we are setting out counterexamples. So we will leave infinite worlds until § 14.6.

EXERCISE 14.3

1. Eliminate the quantifiers in the following formulae for the finite world (i), and for (ii), and calculate their truth values for (i) and for (ii).

   (a) $(\forall x)(Px \supset xRa)$
   (b) $(\forall x)(Px \supset \sim aRx)$
   (c) $(\exists x)[(Px \,\&\, Gx) \,\&\, xRa]$
   (d) $(\forall x)[(Lx \,\&\, Gx) \supset xRa]$
   (e) $(\forall x)(\exists y)\, xRy$
   (f) $(\exists y)(\forall x) \sim xRy$
   (g) $(\forall x)[Px \supset (\exists y)(Py \,\&\, Gy \,\&\, \sim xRy)]$
   (h) $(\forall x)[(Lx \,\&\, (\forall y)(Gy \supset xRy)) \supset xRa]$
   (i) $(\exists y)\, yRa \supset p$
   (j) $(\forall y)(p \supset (\exists x)\, xRy)$

(i)

	P	*G*	*L*
a	1	0	1
b	0	1	0

p
0

R	*a*	*b*
a	0	1
b	1	0

(ii)

	P	*G*	*L*
a	0	1	1

p
1

R	*a*
a	0

14.4 PROPOSITIONS, ARGUMENTS, AND QT-TREES

We begin by considering *QL-forms* and *QL-argument-forms.* QL-forms, just like MQL-forms, are those formulae which contain no propositional constants, and for which there is no dictionary. (See §11.3). The basic definitions for *QT-Necessity, QT-Contradiction* and *QT-Contingency* are essentially the same as for MQT.

Definition: A form is a QT-Necessity iff it is true in every world.

Definition: A form is a QT-Contradiction iff it is false in every world.

Definition: A form is a QT-Contingency iff it is true in some worlds and false in others.

Unfortunately for us, fact A, as set out in §11.3, does not apply in QT. It is possible for every finite expansion of some QL-forms to be true, but for the formula to be false in some infinite world. But fact C can be modified to apply to a restricted set of forms. The modified C is C^+:

C^+ **If** *a form is in Prenex Normal Form*
and ***no existential quantifier is to the left of any universal quantifier,***
and there are ***n*** *universal quantifiers,*
then *the form is a QT-Necessity iff every expansion up to* ***n*** *items or one-item, whichever is larger, is a tautology.*

The following formulae match the description:

$$(\forall x)(\exists y)[Px \supset (Py \vee xRy)],\ (\forall x)(\forall y)(Fx \supset Fy),\ (\forall x)(\exists y)\, xRy,$$
$$(\forall x)(\forall y)(\forall z)(\exists w)(\exists u)(\exists v)[(Fx \,\&\, Hy \,\&\, xRz) \supset ((Fw \vee Hu) \,\&\, \vee\, Sy)]$$

Note that in them all, the universal quantifiers are all to the left of any existentials. The following do not match:

$$(\exists y)(\forall x)\, xRy,\ (\forall x)(\exists y)(\forall z)[(xRy \,\&\, yRz) \supset xRz],\ (\exists x)\, Fx \supset p$$

The method of applying C^+ is the same as for C.

Example: Is $(\forall x)(\exists y)[Px \supset (Py \vee xRy)]$ a QT-Necessity?

1. It does match.
2. There is *one* universal quantifier.
3. Since $[Pa \supset (Pa \vee aRa)]$ is a tautology the formula is a QT-Necessity.

When C^+ cannot be used we have to use some other method of testing QL-forms for QT-Necessity, QT-Contradiction and QT-Contingency. Even when C^+ could be used some other method might be more practical.

Truth-trees are generally applicable to QT. There is a closed truth-tree for every QT-Necessity, and one for every QT-Contradiction. Truth-trees for QT use precisely the same rules as for MQT.

Example: Is $(\exists x)(\forall y)xFy \supset (\forall y)(\exists x)\, xFy$ a QT-Necessity?

1

	√	1.	$\sim[(\exists x)(\forall y)\, xFy \supset (\forall y)(\exists x)\, xFy]$	NF
b	√	2.	$(\exists x)(\forall y)\, xFy$	PC
	√	3.	$\sim(\forall y)(\exists x)xFy$	
a	√	4.	$(\exists y) \sim(\exists x)\, xFy$	3, QN
	√	5.	$\sim(\exists x)\, xFa$	4, EI
a	\	6.	$(\forall y)\, bFy$	2, EI (new constant)
b	\	7.	$(\forall x) \sim xFa$	5, QN
		8.	$\sim bFa$	7, UI
		9.	bFa	6, UI
			X	

So $(\exists x)(\forall y)\, xFy \supset (\forall y)(\exists x)\, xFy$ is a QT-Necessity. Recheck to be sure you can see how each resolution was carried out.

Example: Is $(\forall x)\, xFx \supset (\forall y)(\forall x)\, yFx$ a QT-Necessity?

	√	1.	$\sim[(\forall x)\, xFx \supset (\forall y)(\forall x)\, yFx]$	NF
ba X		2.	$(\forall x)\, xFx$	PC
	√	3.	$\sim(\forall y)(\forall x)\, yFx$	
a	√	4.	$(\exists y) \sim(\forall x)\, yFx$	3, QN
	√	5.	$\sim(\forall x)\, aFx$	4, EI
b	√	6.	$(\exists x) \sim aFx$	5, QN
		7.	$\sim aFb$	6, EI
		8.	aFa	2, UI
		9.	bFb	2, UI
			↑	

Clearly, the tree is not going to close. We can set out a two-item world, (A), as follows

(A)

F	*a*	*b*
a	1	0
b		1

Note that there is one value missing. The value in that space should not matter. But, we leave it blank to see what happens. We expand our form for $[a, b]$ to get:

$$(aFa \,\&\, bFb) \supset ((aFa \,\&\, bFa) \,\&\, (aFb \,\&\, bFb))$$

So: $(1 \,\&\, 1) \supset ((1 \,\&\, \ldots) \,\&\, (0 \,\&\, 1)) = 1 \supset 0 = 0$

Even with the blank space, the formula is false. So (A) is a counterexample. In fact, (A) gives us two counterexamples: (A_1) and (A_2)

A_1

F	*a*	*b*
a	1	0
b	1	1

A_2

F	*a*	*b*
a	1	0
b	0	1

Fact C^+ can be extended to the testing of argument forms. But the argument must meet a *QT short cut condition (QTSCC)*

QTSCC *Every premise and the conclusion must be of Prenex Normal Form.*
In a premise no universal quantifier is to the left of any existential quantifier. In the conclusion no existential quantifier is to the left of any universal quantifier.

Method

1. Check that the argument meets QTSCC.
2. Count the existential quantifers in the premises and add to the number of universal quantifiers in the conclusion.
3. (a) If the total from 2 is one or zero check the one-item expansion only. If valid, the argument-form is valid.
   (b) If the total from 2 is greater than one then check the expansions from one up to that number. If all are valid, the argument-form is valid.

Example: Is $(\forall x)(\forall y)(xFy \supset \sim yFx)$

$\therefore \quad (\forall y) \sim yFy$ valid?

1. It does match QTSCC
2. The total is 1
3. $aFa \supset \sim aFa$

   $\therefore \quad \sim aFa$ is valid, so the argument-form is valid.

Example: Is $(\forall x)(\exists y)\ xFy$

$\therefore (\exists y)(\forall x)\ xFy$ valid?

1. It does *not* match QTSCC. So another method has to be used.

Example: Is $(\forall x)(\forall y)(xFy \supset yFx)$

$\therefore \quad (\forall y)\ yFy$ valid?

1. It does match QTSCC
2. The total is 1.
3. $aFa \supset aFa$

   $\therefore \quad aFa$ is invalid

From step 3 we can easily get the following one-item counterexample:

(B)

F	a
a	0

$aFa \supset aFa \quad = 0 \supset 0 \quad = 1$

$\therefore \quad aFa \quad = 0$

If this short cut method involves worlds which are too large, or if this method cannot be used, we can use truth-trees. Truth-trees are used to test the validity of QT-argument-forms in the same way as for MQT.

Example: Is $(\forall x)(\forall y)(xFy \supset \sim yFx)$

$\therefore \quad (\forall y) \sim yFy$ valid?

a	X	1.	$(\forall x)(\forall y)(xFy \supset \sim yFx)$	P
	√	2.	$\sim(\forall y) \sim yFy$	NC
a	√	3.	$(\exists y) \sim\sim yFy$	2, QN
	√	4.	$\sim\sim aFa$	3, EI
		5.	aFa	4, PC
a	X	6.	$(\forall y)(aFy \supset \sim yFa)$	1, UI
	√	7.	$aFa \supset \sim aFa$	6, UI
			$\sim aFa$ X $\quad \sim aFa$ X	7, PC

So the argument-form is valid.

Example Test the following argument-form for validity.

$$\begin{array}{rl} & (\forall x)\sim(Fx \,\&\, \sim Gx) \\ & (\forall x)([Hx \,\&\, (\exists y)(Gy \,\&\, xKy)] \supset Mx) \\ & (\exists x)[Hx \,\&\, Ox \,\&\, (\exists y)(Fy \,\&\, xKy)] \\ \hline \therefore & (\exists x)(Hx \,\&\, Ox \,\&\, Mx) \end{array}$$

b \	1.	$(\forall x)\sim(Fx \,\&\, \sim Gx)$	P
a \	2.	$(\forall x)([Hx \,\&\, (\exists y)(Gy \,\&\, xKy)] \supset Mx)$	P
a ✓	3.	$(\exists x)[Hx \,\&\, Ox \,\&\, (\exists y)(Fy \,\&\, xKy)]$	P
✓	4.	$\sim(\exists x)(Hx \,\&\, Ox \,\&\, Mx)$	NC
a \	5.	$(\forall x)\sim(Hx \,\&\, Ox \,\&\, Mx)$	4, QN
✓	6.	$Ha \,\&\, Oa \,\&\, (\exists y)(Fy \,\&\, aKy)$	3, EI
	7.	Ha	
	8.	Oa	6, PC
b ✓	9.	$(\exists y)(Fy \,\&\, aKy)$	
✓	10.	$Fb \,\&\, aKb$	9, EI
	11.	Fb	10, PC
	12.	aKb	
✓	13.	$\sim(Ha \,\&\, Oa \,\&\, Ma)$	5, UI
	14.	$\sim Ha$ $\sim Oa$ $\sim Ma$	13, PC
		× ×	
✓	15.	$\sim(Fb \,\&\, \sim Gb)$	1, UI
	16.	$\sim Fb$ Gb	15, PC
		×	
✓	17.	$[Ha \,\&\, (\exists y)(Gy \,\&\, aKy)] \supset Ma$	2, UI
	18.	✓ $\sim[Ha \,\&\, (\exists y)(Gy \,\&\, aKy)]$ Ma	17, PC
		×	
	19.	$\sim Ha$ ✓ $\sim(\exists y)(Gy \,\&\, aKy)$	18a, PC
b \	20.	× $(\forall y)\sim(Gy \,\&\, aKy)$	19b, QN
✓	21.	$\sim(Gb \,\&\, aKb)$	20, UI
	22.	$\sim Gb$ $\sim aKb$	21, PC
		× ×	

The tree closes. ∴ the argument-form is *valid.*

In the trees for MQT, when the tree looked as though it would remain open, we could check to be sure of TUI. This is not always possible in QT. Consider the argument:

$$\begin{array}{rl} & (\forall x)(\exists y)\, xFy \\ \hline \therefore & (\exists y)(\forall x)\, xFy \end{array}$$

The tree goes as follows:

ba	$\backslash$	1.	$(\forall x)(\exists y)\, xFy$	P
	√	2.	$\sim(\exists y)(\forall x)\, xFy$	NC
a	$\backslash$	3.	$(\forall y) \sim(\forall x)\, xFy$	2 QN
	√	4.	$\sim(\forall x)\, xFa$	3 UI
b	√	5.	$(\exists x) \sim xFa$	4 QN
		6.	$\sim bFa$	5 EI
c	√	7.	$(\exists y)\, aFy$	1 UI
d	√	8.	$(\exists y)\, bFy$	1 UI
		9.	aFc	7EI
		10.	bFd	8 EI

Not only is there no closure, but if we try to TUI of either line 1 or line 3, the tree will go on and on for ever, never closing. So far, we are up to four items. We could try out the counterexample suggested by the open tree.

(C)

F	a	b	c	d
a			1	1
b	0			
c				
d				

There are only three values. We could expand the argument for four items. We leave that task to the reader, if you wish.

In fact, we don't need such a large counterexample. All we need is (D):

(D)

F	a	b
a	1	0
b	0	1

$$\begin{array}{lcc} (aFa \vee aFb)\ \&\ (bFa \vee bFb) & = & 1 \\ \hline \therefore\ (aFa\ \&\ bFa) \vee (aFb\ \&\ bFb) & = & 0 \end{array}$$

The argument is clearly invalid.

So, when a tree is clearly going to go on and on to infinity, stop! Look first for a small finite counterexample. If no small finite counterexample is to be found, then it may be necessary to set out a large, even infinite, counterexample. We will discuss infinite counterexamples in §14.6. Note that, unless the completion rule is satisfied, the following principle must be observed:

Counterexamples in QT *must* always be checked (by expansion and substitution of truth-values).

In QT, an open tree does not guarantee a counterexample. This is unlike the situation in either PC or MQT. Nonetheless there is a closed tree for every necessary truth, contradiction, and valid argument in QT. A closed tree is a guarantee, but an open tree is not, so when a counterexample is read off from any open tree it has to be checked.

EXERCISE 14.4

1. Select from the following formulae those which can be dealt with by means of C^+, and work out which of the selected formulae are QT-Necessities by expansion.
    - (a) $(\forall x)(\forall y)(Fx \supset Fy)$
    - (b) $(\forall x)(\exists y)(xRy \supset yRx)$
    - (c) $(\forall x)(Fx \vee Gx)\ .\supset.\ (\forall x)\, Fx \vee (\forall x)\, Gx$
    - (d) $(\forall x)(\exists y)(\exists z)(\forall w)[zRw \supset xRy]$
    - (e) $(\forall z)(\exists x)(\exists y)[(xRy \supset \sim yRx) \supset \sim zRz]$
    - (f) $(\forall x)(\forall y)(\exists z)[zRz \supset (xRy \vee yRx\ .\supset.\ yRy)]$

(g) $(\forall x)(\exists y)(\exists y)[xFx \supset yFz]$
(h) $(\forall x)(\exists y)[(Sy \supset Py) \supset ((Sx \ \& \ Tx) \supset Px)]$
(i) $(\forall x)(\forall y)(\exists z)(\exists w)[xRy \supset yRw]$
(j) $(\exists x)(\exists y)(Fx \supset Fy)$

2. Use truth-trees to test the unselected formulae of Q1 for QT-Necessity.

3. Use truth-trees to show that the following are QT-Necessities.

(a) $(\exists x)(\exists y)\, xFy \equiv (\exists y)(\exists x)\, yFx$
(b) $(\forall x)(\forall y)\, xFy \equiv (\forall y)(\forall x)\, xFy$
(c) $(\forall x)(\forall y)\, xFy \supset (\forall y)\, yFy$
(d) $(\exists x)(\forall y)\, xFy \supset (\exists x)\, xFx$
(e) $(\forall x)[xFx \supset (\exists y)(xFy \supset yFx)]$
(f) $(\forall y)(\forall x) \sim [(\forall z)\, zGx \ \& \sim (\exists w)\, yGw]$
(g) $(\exists x)(\forall y)\, xAy \supset [(\forall x)(\forall y)(yAx \supset yBx) \supset (\forall y)(\exists x)\, xBy]$
(h) $(\exists x) \sim [(\exists y)(Fy \ \& \ xGy) \supset (\exists y)(\sim xGy \supset \sim Hy)] \supset \sim (\forall y)(Fy \supset Hy)$
(i) $(\exists x)(\exists y)(Px \ \& \ Py) \equiv (\exists x)\, Px$
(j) $(\exists x)\, xFx \supset (\exists x)(\exists y)\, xFy$

4. Provide a counterexample for each of the following formulae, and verify your counterexample.

(a) $(\forall x)(\forall y)\, xRy \equiv (\forall x)\, xRx$
(b) $(\forall x)\, xRx \supset (\forall x)(\forall y)(xRy \supset yRx)$
(c) $(\exists x)\, xRx \supset (\forall y)(\exists x)\, yRx$
(d) $(\exists x)(\exists y)\, xRy \supset (\exists x)(\forall y)\, xRy$
(e) $(\exists x)(Fx \ \& \ (\exists y)\, xRy) \supset (\exists x)(Fx \ \& \ xRx)$

5. Select from the following QT-argument-forms those which match the QTSCC, and work out by expansion which are valid and which not, of those selected.

(a) $(\exists x)(\forall y)[Px \ \& \ (Py \supset yRx)],\ (\forall x)(Qx \supset Px) \ / \ \therefore \ (\forall x)(\exists y)[(Qx \ \& \ Tx) \supset (Py \ \& \ xRy)]$
(b) $(\exists x)(Px \ \& \ Mx),\ (\forall x)[Px \supset (\exists y)(Qy \ \& \ xRy)] \ / \ \therefore (\exists x)[Qx \ \& \ (\exists y)(My \ \& \ yRx)]$
(c) $(\forall x)(\forall y)[(Px \ \& \ Py) \supset xRy],\ (\forall x)(\forall y)(\forall z)[(xRy \ \& \ yRz) \supset xSz] \ / \ \therefore \ (\forall x)(\exists y)[Px \supset (Py \ \& \ xSy)]$
(d) $(\exists x)(\forall y)\, xRy \ / \ \therefore \ (\exists x)(\exists y)[xRy \ \& \sim yRx]$
(e) $(\exists x)(\exists y)\, xRy \ \& \ (\exists x)(\exists y)\, xSy \ / \ \therefore \ (\forall x)(\forall y)(xRy \supset xSy)$

6. Use truth-trees to deal with the argument-forms in Q5 which do not match the QTSCC.

7. Use truth-trees to show that the following argument-forms are valid.

(a) $(\exists y)\, yFy \ / \ \therefore \ (\exists y)(\exists x)\, yFx$

(b)
$$\begin{array}{ll} & (\forall x)(\forall y)(xRy \supset \sim yRx) \\ & (\forall x)(Px \supset (\exists y)(Py \ \& \ xRy)) \\ \hline \therefore & \sim (\exists x)(Px \ \& \ (\forall y)(Py \supset yRx)) \end{array}$$

(c)
$$\begin{array}{ll} & (\forall x)(\forall y)(\forall z)((xRy \ \& \ yRz) \supset xRz) \\ & (\forall x)(\forall y)(xRy \supset yRz) \\ \hline \therefore & (\forall x)(\forall y)(xRy \supset xRx) \end{array}$$

(d)
$$\begin{array}{ll} & (\forall x)(\forall y)(xLy \supset xFy) \\ & (\exists x)[Px \,\&\, (\exists y)(xFy \,\&\, {\sim}xLy)] \\ \hline \therefore & {\sim}(\forall x)(\forall y)(xFy \supset xLy) \end{array}$$

(e)
$$\begin{array}{ll} & (\exists x)(Fx \vee Gx) \\ & (\forall x)(Fx \supset {\sim}(\exists y)xHy) \\ & (\forall x)(\forall y)(xHy \equiv (Fx \,\&\, Gy)) \\ \hline \therefore & (\exists x)(\forall y)xHy \supset (\exists x)Gx \end{array}$$

(f)
$$\begin{array}{ll} & (\forall x)(\forall y)(\forall z)[xGy \,\&\, xGz \,.\supset.\, Bxyz] \\ & (\exists x)[Nx \,\&\, (\exists y)(\exists z)(xGy \,\&\, xGz)] \\ \hline \therefore & {\sim}(\forall x)(\forall y)(\forall z)(Bxyz \supset {\sim}Nx) \end{array}$$

(g)
$$\begin{array}{ll} & (\forall x)(Gx \supset (\exists y)(Py \,\&\, xHy)) \\ & (\exists x)(Gx \,\&\, Tx) \,\&\, {\sim}(\forall x)(Gx \supset Tx) \\ & (\forall x)(\forall y)((Gx \,\&\, Tx \,\&\, Gy \,\&\, {\sim}Ty) \supset {\sim}xVy) \\ \hline \therefore & (\exists x)(\exists y)(Px \,\&\, Gy \,\&\, yHx \,\&\, (\forall z)((Gz \,\&\, {\sim}Tz) \supset {\sim}yVz)) \end{array}$$

(h)
$$\begin{array}{ll} & (\forall x)(\forall y)(\forall z)((xFz \,\&\, zFy) \supset xFy) \\ & (\forall x)(\forall y)(\forall z)((zFx \,\&\, zFy) \supset (xFy \vee yFx)) \\ & (\forall x)(\exists y)\, xFy \\ \hline \therefore & (\forall x)(\forall y)((\exists z)(zFx \,\&\, zFy) \supset (\exists z)(xFz \,\&\, yFz)) \end{array}$$

(i)
$$\begin{array}{lc} & p \vee q \vee r \\ & (\exists x)(p \supset {\sim}Fx) \\ & (\forall x)(((\forall x)\, Fx \supset Fx) \supset Fx) \\ & (\forall y)(\forall x)(q \supset xRy) \\ & {\sim}(\forall x)\, xRx \\ \hline \therefore & r \end{array}$$

8. Which of the following are counterexamples to the associated formulae? Provide proof for your answer by the elimination of quantifiers.

(a) $(\forall x)(\forall y)(xRy \supset xRx) \supset (\forall x) {\sim}xRx$

R	a	b
a	1	1
b	0	0

(b) $(\forall x)(\forall y)(xRy \supset yRx) \supset (\forall x)(\exists y)\, xRy$

R	a	b
a	1	0
b	0	0

(c) $(\forall x) {\sim}Px \supset (\exists x)(\forall y)(Px \supset xRy)$

	P
a	0
b	0

R	a	b
a	1	1
b	0	0

(d) $(\forall z) {\sim}zRz \supset (\forall x)(\forall y)(xRy \supset {\sim}yRx)$

R	a	b
a	0	1
b	1	0

(e) $(\forall x)(\forall y)(xRy \supset xRx) \supset (\forall z)\, zRz$

R	a	b
a	1	1
b	0	1

(f) $(\forall x)(\exists y)\, xRy \supset (\exists y)(\forall x)\, xRy$

R	a	b
a	0	0
b	1	0

(g) $(\exists y)(\forall w)((\forall x)\, xRy \supset (\exists v)\, wRv)$

R	a	b
a	1	0
b	0	1

9. Provide a counterexample for each of the following argument-forms, and verify your counterexample by eliminating quantifiers.

(a) $(\exists x)(\exists y)xRy \quad / \quad \therefore\ (\forall x)(\exists y)xRy$

(b) $(\exists x)(\exists y)xRy \quad / \quad \therefore\ (\forall x)(\forall y)xRy$

(c) $(\exists x)(\exists y)xRy \quad / \quad \therefore\ (\exists x)xRx$

(d) $(\forall x)(\exists y)(xRy \,\&\, xSy) \quad / \quad \therefore\ (\exists x)(\forall y)(xRy \,\&\, xSy)$

*(e) $(\exists x)(Fx \supset (\exists y)(p \supset xRy))$
$p \supset (\forall x)Fx$
$(\exists x)(\exists y)(Fx \,\&\, {\sim}Fy \,\&\, xRy)$
$\therefore\ (\exists x)(xRx \supset p)$

14.5 TRANSLATION

We have already seen that (1) is **A** *active*.

Every logician admires Russell. (1)

Let: b = Russell
Lx = x is a logician
xAy = x admires y (note: the active voice)

We translate (1) as (2), using the overall **A** form.

$(\forall x)(Lx \supset xAb)$ (2)

Notice how the quantifier binds an x to the *left* in xAb. Now look at (3) which is **A** *passive* and its translation in (4).

Every logician is admired by Russell. (3)
$(\forall x)(Lx \supset bAx)$ (4)

In the passive we see how the quantifier binds an x to the *right* in bAx. A similar contrast will be found with the **E** *active* and **E** *passive*, **I** *active* and **I** *passive*, **O** *active* and **O** *passive*. For example, compare and contrast (5), (6), (7) and (8).

O *active* Some logician does not admire Russell (5)
$(\exists x)(Lx \,\&\, {\sim}xAb)$ (6)

O *passive* Some logician is not admired by Russell (7)
$(\exists x)(Lx \,\&\, {\sim}bAx)$ (8)

We have already seen that (9) is **A** *over* **A** *active*.

Every logician admires every genius. (9)

Let: $Gx = x$ is a genius.

We can semi-translate (9), showing the overall **A** form, as (10):

$(\forall x)(Lx \supset x$ admires every genius$)$ (10)

We now need to show the **A** form of the consequent, as in (11)

$(\forall x)(Lx \supset (\forall y)(Gy \supset xAy))$ (11)

Note how, since the overall form is **A** *active* the order of x and y about *admires* is the same as the order of the quantifiers which bind them. To translate (12) we would get the same as (11) but with the x and y in opposite order to the quantifiers, see (13).

A *over* **A** *passive* Every logician is admired by every genius. (12)
$(\forall x)(Lx \supset (\forall y)(Gy \supset yAx))$ (13)

Compare and contrast (14), (15), (16) and (17).

I *over* **E** *active* Some logician admires no genius. (14)
$(\exists x)(Lx \;\&\; (\forall y)(Gy \supset \sim xAy))$ (15)

I *over* **E** *passive* Some logician is admired by no genius. (16)
$(\exists x)(Lx \;\&\; (\forall y)(Gy \supset \sim yAx))$ (17)

We have also seen that (18) is **A** *by* **A** *active active*

Every logician who respects everyone admires Russell. (18)

To translate (18) we need to show the **A** form in the antecedent. We semi-translate (18) as (19).

$(\forall x)(Lx$ and x respects every person $\supset xAb)$ (19)

We then get (20) if we let: $Px = x$ is a person.
$xRy = x$ respects y.

$(\forall x)[(Lx \;\&\; (\forall y)(Py \supset xRy)) \supset xAb]$ (20)

(21) is **A** *by* **I** *passive active*

Every logician who is respected by someone admires Russell. (21)

So we translate to (22):

$(\forall x)[(Lx \;\&\; (\exists y)(Py \;\&\; yRx)) \supset xAb]$ (22)

Consider (23), which we earlier described as **A** *by* **I** *active over* **A** *active*:

Every logician who respects someone admires every writer. (23)

If we let $Wx = x$ is a writer, then (23) translates to (24):

$(\forall x)[(Lx \;\&\; (\exists y)(Py \;\&\; xRy)) \supset (\forall z)(Wz \supset xAz)]$ (24)

Compare (23) and (24) with (25) and (26)

A *by* **I** *passive over* **A** *passive*:

Every logician who is respected by someone is admired by every writer (25)
$(\forall x)[(Lx \;\&\; (\exists y)(Py \;\&\; yRx)) \supset (\forall z)(Wz \supset zAx)]$ (26)

See how the change to passive reverses the order of variables.

Now we will look at a case where a *triadic* predicate is involved. We let

$Lxyz$ = x loves y at z
Tx = x is a time.

It seems reasonable to describe (27) as **A** *over* **I** *over* **I** *active*

Everybody loves somebody sometime (27)

(27) translates to (28).

$$(\forall x)[Px \supset (\exists y)(Py \; \& \; (\exists z)(Tz \; \& \; Lxyz)] \tag{28}$$

Similarly it seems reasonable to say that (29) is **A** *over* **I** *over* **I** *passive*, and it translates as (30).

Everybody is loved by somebody sometime. (29)

$$(\forall x)[Px \supset (\exists y)(Py \; \& \; (\exists z)(Tz \; \& \; Lyxz))] \tag{30}$$

This use of "active" and "passive" for predicates of greater adinity than two is risky, but can sometimes be helpful.

Sometimes when translating **E** propositions it can be helpful to consider the **E** in terms of a *negated* **I**. Consider (31).

No-one loves anyone (31)

We can re-express this as (32)

Not even one person loves at least one person (32)

The change to negated **I** forces clarification of "anyone". The translation is then of a negated **I** *over* **I** *active*, to give (33), if $xLy = x$ loves y

$$\sim(\exists x)[Px \; \& \; (\exists y)(Py \; \& \; xLy)] \tag{33}$$

It is important, at this stage, to re-emphasize the difference between (34) and (35).

$$(\forall x)(\exists y) \; xRy \tag{34}$$
$$(\exists y)(\forall x) \; xRy \tag{35}$$

In §14.3 (34) and (35) were discussed in a restricted universe of persons. (34) presents a picture of each person respecting someone, but not necessarily the same person. (35) presents a picture of each person respecting the same (at least one) person. Outside a restricted universe of persons we would need (36) and (37) respectively.

$$(\forall x)(Px \supset (\exists y)(Py \; \& \; xRy)) \tag{36}$$
$$(\exists y)(Py \; \& \; (\forall x)(Px \supset xRy)) \tag{37}$$

(36) is **A** *over* **I** *active* while (37) is **I** *over* **A** *passive*.

Let: $xCy = x$ is a cousin of y.

(38) is symbolized by (39). Note how "any two" indicates two universal quantifiers.

For any two people, if the first is a cousin of the second, then the second is a cousin of the first. (38)

$$(\forall x)(\forall y)[(Px \; \& \; Py) \supset (xCy \supset yCx)] \tag{39}$$

If the universe of discourse were restricted to persons we would get (40)

$$(\forall x)(\forall y)(xCy \supset yCx) \tag{40}$$

Let: $xTy = x$ is taller than y.

(41) is symbolized by (42), or by (43) in a restricted universe of discourse. In this case "any three" indicates three universal quantifiers.

For any three people, if the first is taller than the second, and the second is taller than the third, then the first is taller than the third. (41)

$$(\forall x)(\forall y)(\forall z)[(Px \,\&\, Py \,\&\, Pz) \supset ((xTy \,\&\, yTz) \supset xTz))] \tag{42}$$

$$(\forall x)(\forall y)(\forall z)[(xTy \,\&\, yTz) \supset xTz] \tag{43}$$

EXERCISE 14.5

In questions 1-7 translate into QL using only the dictionary provided.

1\.
e .... English
m .... Maths
s .... Science
Sx .... x is a subject
xHy .... x is harder than y

(a) English is a subject.
(b) Maths, Science and English are all subjects.
(c) Science is harder than Maths.
(d) Neither English nor Maths is harder than Science.
(e) Science is a harder subject than Maths.
(f) There is a subject which is harder than both Maths and English.
(g) No subject is harder than Science.
(h) At least one of Science and English is harder than Maths.
(i) Any subject harder than Science is also harder than English.
*(j) A few subjects are harder than Maths but few subjects are harder than Science.
*(k) Only a subject harder than Maths can qualify as being a harder subject than English.

2\.
b = Bob
c = Carol
Px = x is a person
Gx = x is generous
xRy = x is related to y

(a) Everyone is related to Bob.
(b) Someone is related to Bob.
(c) No-one is related to Bob.
(d) Someone is not related to Bob.
(e) Everyone who is related to Bob is generous.
(f) Everyone who is not related to Bob is generous.
(g) Some one who is related to Bob is not generous.
(h) Some one who is generous is not related to Bob.
(i) Everyone who is generous is related to Bob.
(j) Everyone who is related to Carol is related to Bob.
(k) Everyone who is related to Carol is either generous or related to Bob.

3. b = Betty c = Carl d = David
Px = x is a person Kx = x is kind
xHy = x helps y Dx = x is deserving

(a) Betty helps Carl but Carl does not help David.
(b) Everyone helps Carl.
(c) Everyone is helped by Carl.
(d) Everyone helps someone.
(e) Everyone is helped by someone.
(f) Everyone who helps David is kind.
(g) Everyone who is helped by David is deserving.
(h) Everyone who helps someone is kind.
(i) Everyone who is helped by someone is deserving.
(j) Every kind person helps someone.
(k) Every deserving person is helped by someone.

4. *Universe* = persons j = Joe Blow
b = Bob m = Mal
d = Dame Edna Px = x is a politician
xCy = x convinces y xLy = x loves y

(a) Bob is not convinced by Mal.
(b) Bob convinces someone.
(c) Not everyone is convinced by Mal.
(d) No politician convinces Joe Blow.
(e) Dame Edna loves all politicians.
(f) Dame Edna is not convinced by any politician.
(g) Politicans convince no-one.
(h) Someone who is not a politician loves someone who is.
(i) Although Mal convinces Joe Blow, not every politician does.

5. a .... Lord Alpha
e .... the evil warlock
m .... the fair maiden
Px .... x is a person
Gx .... x is good
xIy .... x imprisons y in a dungeon
xLy .... x loves y
$Rxyz$.... x rescues y from z

(a) The evil warlock does not love the fair maiden, and has imprisoned her in a dungeon.
(b) If the evil warlock loves nobody then he doesn't love himself.
* (c) Anyone who loves everyone is good.
* (d) The evil warlock will imprison himself in a dungeon if anyone rescues the fair maiden from him.
(e) Lord Alpha, who loves and is loved by the fair maiden, rescues her from the evil warlock.

6. a .... Alf
b .... Betty
c .... Colin

Hx .... x is a person
Mx .... x is a male
xPy .... x is a parent of y

(a) Alf is a parent of Betty.
(b) Someone is not a parent of Betty.
(c) Alf is Betty's father.
(d) Alf has a son.
(e) Betty is a female person.
(f) Betty is Alf's daughter.
(g) Betty has a parent.
(h) Colin is a grandparent of Betty.
(i) Betty has a grandparent.
*(j) Betty has a grandmother.
(k) Everyone has a parent.
*(l) Not every parent is a male.
*(m) Fathers are all male.
(n) No person is his own parent.

7. Supply your own dictionary and translate the following into QL.

(a) Bertrand Russell was a philosopher.
(b) All philosophers are persons.
(c) Confucius was a philospher only if K'ung Tzŭ was.
(d) There is someone whom everyone loves.
(e) Everyone loves at least one person.
(f) Nobody loves nobody.
(g) There is a man who loves himself.
(h) If anyone was a philospher, Bertrand Russell was.
*(i) Logicians and philosophers are both interesting and intelligent.
(j) No mammoths exist.
(k) Some people hear but do not listen.
(l) Only those who listen are worth listening to.
*(m) Laziness never leads to success.
*(n) Laziness sometimes is damaging.
*(o) Nothing but a miracle can save us now.
*(p) Nobody save a genius would have thought of that.
(q) All cheerful, hardworking students with a big work load deserve a rest.

In questions 8-10 you are given formulae which should be translated into English using the dictionary provided.

8. Px .... x is a person
Sx .... x is a sport
Mx .... x is a mental activity
xGy .... x is good at y

(a) $\sim(\forall x)(Sx \vee Mx)$

(b) $(\forall x)[Px \supset (\exists y)(Sy \,\&\, xGy)]$

*(c) $\sim(\forall x)[Px \supset (\forall y)(My \supset xGy)]$

*(d) $(\exists x)[\sim(Sx \vee Mx) \,\&\, (\forall y)(Py \supset yGx)]$

9\.
Px .... x is an adult person
Mx .... x is male
Fx .... is female
xLy .... x loves y

(a) $\sim(\forall x)(Px \supset Mx)$

(b) $(\forall x)[(Px \,\&\, Mx) \supset (\exists y)(Py \,\&\, Fy \,\&\, yLx)]$

(c) $(\exists x)(\exists y)(Px \,\&\, Py \,\&\, xLy \,\&\, \sim yLx)$

*(d) $(\exists x)[Mx \,\&\, xLx \,\&\, (\forall y)(Fy \supset \sim xLy)]$

*(e) $(\forall x)[Px \supset (Mx \not\equiv Fx)] \supset \sim(\exists x)(Px \,\&\, Mx \,\&\, Fx)$

10\.
Rx .... x is a real number
Px .... x is a positive number
Nx .... x is a negative number
xGy .... x is greater than y
xIy .... x equals y

(a) $(\forall x)(Px \supset Rx)$

(b) $(\exists x)[Rx \,\&\, \sim(Px \vee Nx)]$

*(c) $(\forall x)(\forall y)[(Rx \,\&\, Ry) \supset (xGy \vee yGx \vee xIy)]$

(d) $(\forall x)[Px \supset \sim(\exists y)(Ny \,\&\, yGx)]$

*(e) $(\forall x)[Rx \,\&\, (\exists y)(Py \,\&\, xGy) \,.\!\supset .\, Px]$

14.6 INFINITE WORLDS AND COUNTEREXAMPLES

Our main problem with infinite worlds is the problem of counterexamples. For example, consider argument form (1), which certainly does not match *QTSCC.*

$$\begin{array}{l} (\forall x)(\exists y)\, xSy \\ (\forall x)(\forall y)(\forall z)((xSy \,\&\, ySz) \supset xSz) \\ (\forall x) \sim xSx \\ \hline \therefore\ (\exists x)(\forall y)\, ySx \end{array} \tag{1}$$

The tree begins as follows:

b	\	1.	$(\forall x)(\exists y)\, xSy$	P
		2.	$(\forall x)(\forall y)(\forall z)((xSy \,\&\, ySz) \supset xSy)$	P
		3.	$(\forall x) \sim xSx$	P
	√	4.	$\sim(\exists x)(\forall y)\, ySx$	NC
a	\	5.	$(\forall x) \sim(\forall y)\, ySx$	4, QN
	√	6.	$\sim(\forall y)\, ySa$	5, UI
b	√	7.	$(\exists y) \sim ySa$	6, QN
		8.	$\sim bSa$	7, EI
c	√	9.	$(\exists y)\, bSy$	1, UI
		10.	bSc	9, EI

The tree will not close. Attempts for TUI will lead to an infinite open path. So we might try for a finite counterexample in a two-item world by using PC methods (e.g. MAV, or trees, or even tables). The argument expands to (2).

$$\begin{array}{lll}
 & (aSa \lor aSb) \mathbin{\&} (bSa \lor bSb) & \text{P} \\
 & ((aSa \mathbin{\&} aSa) \supset aSa) \mathbin{\&} ((bSa \mathbin{\&} aSa) \supset bSa) \mathbin{\&} & \\
 & ((aSb \mathbin{\&} bSa) \supset aSa) \mathbin{\&} ((bSb \mathbin{\&} bSa) \supset bSa) \mathbin{\&} & \\
 & ((aSa \mathbin{\&} aSb) \supset aSb) \mathbin{\&} ((bSa \mathbin{\&} aSb) \supset bSb) \mathbin{\&} & \\
 & ((aSb \mathbin{\&} bSb) \supset aSb) \mathbin{\&} ((bSb \mathbin{\&} bSb) \supset bSb) & \text{P} \\
 & {\sim} aSa \mathbin{\&} {\sim} bSb & \text{P} \\
\hline
\therefore & (aSa \mathbin{\&} aSb) \lor (bSa \mathbin{\&} bSb) & \qquad (2)
\end{array}$$

The expansion of the second premise can be shortened because six of the eight conjuncts are tautologies. So we get (3)

$$\begin{array}{lll}
 & (aSa \lor aSb) \mathbin{\&} (bSa \lor bSb) & \\
 & ((aSb \mathbin{\&} bSa) \supset aSa) \mathbin{\&} ((bSa \mathbin{\&} aSb) \supset bSb) & \\
 & {\sim} aSa \mathbin{\&} {\sim} bSb & \\
\hline
\therefore & (aSa \mathbin{\&} aSb) \lor (bSa \mathbin{\&} bSb) & \qquad (3)
\end{array}$$

It will be discovered that (3) is valid. Indeed the expanded premises are inconsistent. This can be shown in a tree. We leave that task to the reader.

In fact, this will be so in every finite world. Hence, there is no counterexample in any finite world.

We can *interpret* argument-form (1) in the following way.

Let: *Universe* = numbers
xSy = x is smaller than y.

The interpretation, in English is (4).

Every number is smaller than some number or other.
If one number is smaller than a second and the second is smaller than a third, then the first is smaller than the third.
No number is smaller than itself.

Hence, there is a number than which all numbers are smaller. (4)

In *finite* possible worlds of numbers the premises are inconsistent. But in any *infinite* possible world of natural numbers, 1, 2, 3, ... , the premises are all true, but the conclusion false. We have *interpreted* the symbolization of the argument-form by fixing a dictionary. If the form is a valid form, it will be valid for every interpretation.

The worlds of natural numbers are most useful for providing infinite counterexamples. They may also be used for finite counterexamples as well. A certain amount of inventiveness is needed in order to pick appropriate properties and relations for the interpretation in the dictionary. It must be quite clear that, under the interpretation, the premises are true and the conclusion false. For example, consider argument-form (5).

$$\begin{array}{ll}
 & (\forall x)(Ax \supset Bx) \\
 & (\forall x)(Cx \supset Bx) \\
\hline
\therefore & (\forall x)(Ax \supset Cx)
\end{array} \qquad (5)$$

We can provide an infinite counterexample by the following interpretation:

Let: *Universe* = numbers
Ax = x is divisible by 6
Bx = x is divisible by 2
Cx = x is divisible by 4.

Under interpretation, (5) is (6)

Every number divisible by 6 is divisible by 2
Every number divisible by 4 is divisible by 2
∴ Every number divisible by 6 is divisible by 4 (6)

The premises are true, the conclusion false. If we altered this interpretation of (5) by:

Let: *Universe* = natural numbers 1 to 20

then we would have a finite counterexample.

This method of counterexamples by interpretation can be extended to the testing of arguments. In that case the counterexample is called a counterargument. It must be remembered that an infinite counterexample cannot be checked by expansion and calculation. So the interpretation must show clearly that formulae have the appropriate truth-value without calculation.

The following steps should be followed when developing an infinite counterexample.

1. State what the universe of discourse is, (e.g. the natural numbers: 1, 2, 3, ...)
2. Give each predicate, monadic, dyadic, etc. an interpretation relevant to the universe, (e.g. $Ex = x$ is even, $xLy = x$ is less than y)
3. Make each individual constant the name of some one thing in the restricted universe (e.g. $a = 0$, $b = 1$)
4. Make each propositional letter express some proposition which is clearly true or false. (e.g. for false: P = odd numbers are divisible by 2).
5. Write down the reading of each formula in accordance with 1 to 4, and show what its truth-value is.

Free occurrences of individual variables should be dealt with by universal closure.

EXERCISE 14.6

1. Translate the following arguments into QL, using only the dictionaries provided. Test them for validity in QT, using the method of your choice.
   (a) There is at least one proposition implied by every proposition. Since contradictions are all propositions, every contradiction implies some proposition or other.
   ($Px = x$ is a proposition; $Cx = x$ is a contradiction; $xIy = x$ implies y)
   (b) If anyone is superior to anyone, then the latter is not superior to the former. Hence, no-one can be superior to himself.
   ($Px = x$ is a person; $xSy = x$ is superior to y.)
   (c) All gifts are given by someone or other. Although some gifts are tangible, not all are. Gifts which are tangible are not as valuable as gifts which are not tangible. So some people give gifts which are not as valuable as gifts which are intangible.
   ($Px = x$ is a person; $Gx = x$ is a gift; $xHy = x$ gives y; $Tx = x$ is tangible; $xVy = x$ is a valuable as y).

(d) Every word fiend is a scrabble player. Crossword checkers are all word fiends. Since there is at least one book which every scrabble player uses, it follows that crossword checkers all use at least some book or other.
($Wx = x$ is a word fiend; $Sx = x$ is a scrabble player; $Cx = x$ is a crossword checker; $Bx = x$ is a book; $xUy = x$ uses y)

(e) Every philosophical empiricist admires Hume. Some philosophical idealists like no one who admires Hume. Therefore, some philosophical idealists like no philosophical empiricist.
($Ex = x$ is a philosophical empiricist; $Ix = x$ is a philosophical idealist; $xLy = x$ likes y; $xAy = x$ admires y; $a =$ Hume).

(f) Every major newspaper is biased against the Communist party. If such a newspaper is biased against a political party then it is not biased in favour of it. Since the Communist party is a political party it follows that there is at least one political party which major newspapers are not biased in favour of.
($c =$ the communist party; $Mx = x$ is a major newspaper; $Px = x$ is a political party; $xBy = x$ is biased against y; $xFy = x$ is biased in favour of y).

(g) Vacant allotments provide no income for their owners. Any owner of real estate must pay rates on it. Therefore any owner of a vacant allotment must pay rates on something which provides no income for its owner.
($Vx = x$ is a vacant allotment; $Rx = x$ is real estate; $xIy = x$ provides income for y; $xPy = x$ pays rates on y; $xOy = x$ owns y.)

(h) Every tautology is implied by every tautology, contingency and contradiction. Tautologies, contingencies and contradictions are all propositions. Since there is at least one tautology, it follows that every contradiction implies some proposition or other.
($Px = x$ is a proposition; $Tx = x$ is a tautology; $Sx = x$ is a contingency; $Cx = x$ is a contradiction; $xIy = x$ implies y.)

(i) Left wing reformers upset people. Right wing conservatives are certainly not left wing people. So right wing conservatives do not upset people.
($Px = x$ is a person; $Lx = x$ is a left wing reformer; $xUy = x$ upsets y; $Rx = x$ is a right wing conservative.)

(j) A substance is unlimited by anything. Any substance would be limited only by a different substance. If any two substances are different from one another, then one limits the other and is limited by the other. Consequently, if any substance exists then no substance is different from it.
($Sx = x$ is a substance; $xLy = x$ limits y; $xLy = x$ is different from y.)

(k) Every citizen is either a patriot or a traitor. Patriots, and only patriots, are honoured by governments. Some citizens are honoured by governments, and so it follows that some citizens are traitors.
($Cx = x$ is a citizen; $Px = x$ is a patriot; $Tx = x$ is a traitor; $xHy = x$ honours y; $Gx = x$ is a government.)

(l) All land is owned by someone or other. Some land is owned by the Crown. All land not owned by the Crown is privately owned. Hence, some land is privately owned.
($c =$ the Crown; $Lx = x$ is land; $Px = x$ is a person; $Rx = x$ is privately owned; $xOy = x$ owns y.)

(m) Salesmen will talk to people they know well. Salesman do not talk to any person who is not a prospective buyer. So, only prospective buyers are people salesmen know well.
($Px = x$ is a person; $Sx = x$ is a salesman; $Bx = x$ is a prospective buyer; $xKy = x$ knows y well; $xTy = x$ will talk to y.)

(n) If the report is accepted then some recommendations in the report will be put into effect and some will not. All the recommendations in the report would be put into effect if we were to have real progress. So, we will not have real progress, even though some recommendations in the report will be put into effect.
(a = the report; W = We will have real progress; $Ax = x$ is accepted; $Rx = x$ is a recommendation; $Px = x$ is put into effect; $xIy = x$ is in y.)

(o) If some books should be censored then all books should be censored. Why? Because a book should be censored only if there is at least one person who knows that it is certain to corrupt and deprave. Any person who knows that some book is certain to corrupt and deprave will be infallible. But no one is infallible.
($Bx = x$ is a book; $Cx = x$ should be censored; $Px = x$ is a person; $Ix = x$ is infallible; $xKy = x$ knows that y is certain to corrupt and deprave.)

(p) Every new product has some defect or other. No careful buyer wants to purchase any product with any defect. So, careful buyers do not want to purchase new products.
($Px = x$ is a product; $Nx = x$ is new; $Dx = x$ is a defect; $Cx = x$ is a careful buyer; $xHy = x$ has y; $xWy = x$ wants to purchase y)

(q) Whoever belongs to the Jet set is richer than any member of the Golf Club. Not everyone who belongs to the Jet set is richer than everyone who does not belong. So there is someone who is richer than everyone in the Golf Club.
($Px = x$ is a person; $Jx = x$ belongs to the Jet set; $Gx = x$ is a member of the Golf Club; $xRy = x$ is richer than y)

(r) If someone moves a motion and that motion is ruled out of order, then that motion lapses only if there is no successful challenge. Mr Brown moved a motion which was ruled out of order, but there was a successful challenge. So, at least one of Mr Brown's motions did not lapse.
($Px = x$ is a person; $Rx = x$ is a motion; $xMy = x$ moves y; $Qx = x$ is ruled out of order; $Lx = x$ lapses; Q = there was a successful challenge; b = Mr Brown.)

(s) Bernie and Alice are people. If Bernie is married to Alice then either Alice has five sons or she has five daughters (but not both). Now anyone who knows Bernie and Alice knows that they are married. Moreover, Alice has five daughters only if she also has five sons; and she has five sons only if she is an Australian. It so happens that Bernie and Alice are known by Norma (who is a wonderful person). Now it stands to reason that Bernie and Alice are married provided that someone knows they're married. From all this it follows that Alice is Australian, and has five sons but does not have five daughters.
(a = Alice; b = Bernie; $Px = x$ is a person; $xMy = x$ is married to y; $Sx = x$ has five sons; $Dx = x$ has five daughters; $xKy = x$ knows y; $Tx = x$ knows that Bernie and Alice are married; $Ax = x$ is Australian; n = Norma; $Wx = x$ is wonderful.)

2. Translate the following arguments into QL, setting out your dictionary clearly, and then test for validity in QT, using the method of your choice.

(a) All very intelligent people manage to pass the exam. Alan is a student, but he is not very bright. Hence Alan will fail the exam.

(b) No number is greater than itself. Thus no number is greater than every number.

(c) If black holes exist then some objects are invisible. Black holes have strong gravitational fields. Hence some invisible objects have strong gravitational fields.

(d) Any belief is either rational, irrational (against reason), or extra-rational (outside reason). Any irrational belief is not rational. Alan's belief in the transcendent is

an extra-rational belief. Therefore Alan's belief in the transcendent is not justified, since any irrational belief is unjustified.
(Symbolize Alan's belief in the transcendent by an individual constant.)

(e) Some males love all females. Hence any particular female is loved by at least one male.

(f) Was the Hatter mad in assessing the following argument as invalid?

> I see what I eat. Therefore I eat what I see.
>
> (Lewis Carroll, *Alice in Wonderland*)

(g) Anyone who studies logic is sensible. All sensible people should be congratulated. Hence anyone who studies logic deserves congratulations. (Hint: the solution is easier if you restrict the universe to people; if you don't you will have to include "*Px* ... *x* is a person" in your dictionary.)

(h) K'ung Tzǔ was Chinese, and Mo Tzǔ was his intellectual opponent. All intellectual opponents of Chinese were Chinese, so Mo Tzǔ was Chinese.

(i) No beer tastes better than Fourex. It is clear then that Bloopo doesn't taste better than Fourex, since Bloopo is a beer.

(j) Since Mars is a red planet, any astronaut that lands on Mars lands on a planet.

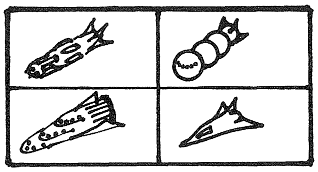
(k) Anyone who teaches Chemistry or organizes sporting activities works hard. Now Ivor has taught Chemistry provided only that he has also organized sporting activities. But the general claim that any person who teaches Chemistry also organizes sport is a false claim. Hence Ivor has not worked hard.

(l) Some people are older than others. Alice is a person and so is Bernie. Hence either Alice is older than Bernie or Bernie is older than Alice.

(m) Anyone who gains satisfaction does not despair. Now anyone who gets this problem out will gain satisfaction. Therefore, no matter what person it is, unless that person gets this problem out that person will despair.

(n) Given any two numbers, if the first is greater than the second then the second is not greater than the first. Thus no number is greater than itself.

Puzzle 14 A sector of space is charted into four quadrants: A51, A52, A53 and A54. In each quadrant is a Rhul spaceship. There is a scout, a tanker, a heavy cruiser and a destroyer. The ships each have a different battle status. One is at maximum efficiency (status I), one is retiring (status II), one has a new crew (status III) and one is damaged (status IV). From the following decoded messages work out which ship is in what status in which sector.



1. The cruiser has a better status than the scout.
2. The tanker is in sector A52.
3. The destroyer has just left A54.
4. The damaged ship is in A53.
5. The destroyer has a new crew.
6. The ship in A54 is retiring.

14.7 SUMMARY

In addition to the *properties* (monadic predicates) considered by MQT, standard *Quantification Theory (QT)* deals with *relations* (polyadic predicates). The language *QL* is formed from MQL by replacing monadic predicate letters with n-adic predicate letters: the *adinity* (n) of these letters equals the number of associated individual letters, standardly written in the suffix position but in the case of dyadic predicates often written on either side e.g., *Fx, Lab* or *aLb, Gxyz.*

Dyadic relations in a given world may be depicted on world-diagrams using arrows or slashed-arrows to indicate whether a relation does or doesn't hold between an ordered pair of items. They may also be displayed by means of Cayley tables. Relations of higher adinity may be displayed in single column tables with the matrix listing the various item sequences. We use "[*a, ..., b*]" as an abbreviation for "any world whose items are *a, ..., b*".

With respect to any world: $(\forall x)\alpha$ is true iff every itemization of α is true; $(\exists x)\alpha$ is true iff at least one itemization of α is true. *Modal* properties and relations (including validity) in QT are defined as for MQT, with "QT" substituted for "MQT". Unlike MQL-forms, it is possible for a QL-form to be true in all finite worlds but false in an infinite world. If a QL-form is in PNF with n $\forall$s and no $\exists$ preceding any $\forall$, then it is a *QT-Necessity* iff every expansion up to n items (or one item if $n = 0$) is a tautology.

If a QL-argument-form meets the *QT Short Cut Condition (QTSCC:* each premise and the conclusion are in PNF, in each premise no $\forall$ precedes any $\exists$, and in the conclusion no $\exists$ precedes any $\forall$), *then* add the number of $\exists$s in the premises to the number of $\forall$s in the conclusion to give a total of t: if $t = 0$ or 1 then the argument-form is valid iff its one-item expansion is valid; otherwise the argument-form is valid iff its expansions up to t-items are valid.

QT-trees work the same way as MQT-trees. Sometimes however, it will be impossible to TUI a formula, and consequently the tree can never be completed. In such cases we can try to find a finite counterexample (using an open path as a hint) and test this by substitution. Sometimes, the only counterexamples will be infinite. One way of generating an infinite counterexample for a QL-argument-form is to choose the universe to be the set of natural numbers, and interpret the individual and predicate letters in terms of numbers and mathematical predicates which clearly make the premises true and the conclusion false. Unlike PC and MQT, there can be no decision procedure for QT in general.

In *translating* from English to QL, it is useful to distinguish between the *active* voice (e.g., All cats like mice) and the *passive* voice (e.g., All cats are liked by mice). If we choose a predicate in the active voice, the passive voice can be expressed by changing the *order* of the individual letters around the predicate e.g., using "*xLy*" for "*x* loves *y*" we may express "*x* is loved by *y*" as "*yLx*".

The letters "A", "E", "I", "O" may be used to classify phrases as well as sentences:

type	*sentence*	*phrase*
A	Every *A* is *B*	Every *A*
E	No *A* is *B*	No *A*
I	Some *A* is *B*	Some *A*
O	Some *A* is not *B*	Not all *A*

Using "by" and "over" as abbreviations for "qualified by" and "with scope over", many sentence constructions may now be classified e.g.,

> "Every cat who likes no mice is liked by some mice."

is **A** by **E** active over **I** passive. In this notation the first letter (here "**A**") and the last word (here "passive") give the overall form. Most propositions can be expressed in English sentences fitting this classification scheme.

Translation from English into QL is best done in a *top-down* fashion. For instance the above example may be translated in stages as follows:

$(\forall x)$ [x is a cat who likes no mice $\supset$ x is liked by some mice]
$(\forall x)$ [(Cx & x likes no mice) $\supset$ some mice like x]
$(\forall x)$ [(Cx & $\sim x$ likes some mice) $\supset (\exists y)\,(My\ \&\ yLx)$]
$(\forall x)\ [(Cx\ \&\ \sim(\exists y)\,(My\ \&\ xLy)) \supset (\exists y)\,(My\ \&\ yLx)]$

The order of mixed quantifiers is important, e.g., $(\forall x)\,(\exists y)\,\alpha$ is *not* equivalent to $(\exists y)\,(\forall x)\,\alpha$.

15

Natural Deduction For QT

15.1 SUBSTITUTION RULES

The system of natural deduction which was set out in Chapter 8 can be extended to QT. All the inference and substitution rules of PC are included in the system for QT. Rules are added for quantification.

The rules added for quantifiers are two substitution rules and four inference rules. The substitution rules and two of the inference rules are very straightforward. We deal with them first. We then look at the two more difficult inference rules.

The Substitution Rules are simply the Quantifier Negation principles. We read '$::$' as 'can replace or be replaced by'. ν is any individual variable.

$\sim(\forall\nu)\alpha \quad :: \quad (\exists\nu)\sim\alpha$ QN

$\sim(\exists\nu)\alpha \quad :: \quad (\forall\nu)\sim\alpha$ QN

Using these substitution rules and the PC rules we can demonstrate the validity of (1)

$$\frac{\sim(\forall x)(Fx \supset Gx)}{\therefore\ (\exists x)(Fx \;\&\; \sim Gx)} \qquad (1)$$

1.	$\sim(\forall x)(Fx \supset Gx)$	P / $\therefore (\exists x)(Fx \;\&\; \sim Gx)$
2.	$(\exists x)\sim(Fx \supset Gx)$	1 QN
3.	$(\exists x)\sim(\sim Fx \vee Gx)$	2 MI
4.	$(\exists x)(\sim\sim Fx \;\&\; \sim Gx)$	3 DeM
5.	$(\exists x)(Fx \;\&\; \sim Gx)$	4 DN

EXERCISE 15.1

1. Construct deductions to show that the following are valid.

   (a) $(\exists x)(Fx \;\&\; Gx)$ / $\therefore \sim(\forall x)(Fx \supset \sim Gx)$

   (b) $(\forall x)(Fx \supset \sim(\exists y)(Gy \;\&\; xRy))$ / $\therefore \sim(\exists x)(Fx \;\&\; \sim(\forall y)(Gy \supset \sim xRy))$

   (c) $(\forall x)(Fx \supset Gx)$ / $\therefore (\forall x)(\sim Gx \supset \sim Fx)$

   (d) $(\forall x)(Fx \supset \sim Gx)$ / $\therefore (\forall x)(Gx \supset \sim Fx)$

   (e) $(\forall x)\,Fx$ / $\therefore \sim(\exists x)\sim Fx$

15.2 UNIVERSAL INSTANTIATION AND EXISTENTIAL GENERALIZATION

Universal Instantiation (UI) for natural deduction is exactly the same as UI in truth-trees.

(UI) $(\forall \nu)\, \varphi\nu$

$\therefore\ \varphi\kappa$ where $\varphi\kappa$ is an itemization of $\varphi\nu$ to *any* individual constant, κ.

With this rule we show (1) to be valid

$$\begin{array}{ll} & (\forall x)(Fx \supset Gx) \\ & \underline{\qquad Fa \qquad} \\ \therefore & \quad Ga \end{array} \tag{1}$$

1.	$(\forall x)(Fx \supset Gx)$	P
2.	Fa	P / $\therefore\ Ga$
3.	$Fa \supset Ga$	1 UI
4.	Ga	2, 3 AA

The second simple rule is Existential Generalization (EG). The rule is best set out using the notation: $\alpha\,(\nu//\mathrm{s})$ where s is any individual constant or variable and $\alpha\,(\nu//\mathrm{s})$ is the result of substituting ν for *every* occurrence of s in α

For example:	$Fa\,(x \,//\, a)$	is Fx
	$(Fb \supset Gb)(y \,//\, b)$	is $(Fy \supset Gy)$
	$(\exists y)\, Fa\,(y \,//\, a)$	is $(\exists y)\, Fy$

Now we set out EG:

EG α

$\therefore (\exists \nu)\alpha\,(\nu \,//\, \kappa)$ where κ is an individual constant, and ν does not occur in α

Some examples of this rule would be

$$\begin{array}{cccc} Fa & Fb \vee Gb & aFb & cFc \\ \therefore (\exists x)\, Fx & \therefore\ (\exists y)(Fy \vee Gy) & \therefore\ (\exists y)\, aFy & \therefore\ (\exists z)\, zFz \end{array}$$

We use the rules we have to show (2) valid

$$\begin{array}{ll} & (\forall x)(Fx \supset Gx) \\ & \underline{\qquad Fa \qquad} \\ \therefore & (\exists x)\, Gx \end{array} \tag{2}$$

1.	$(\forall x)(Fx \supset Gx)$	P
2.	Fa	P / $\therefore (\exists x)\, Gx$
3.	$Fa \supset Ga$	1 UI
4.	Ga	2, 3 AA
5.	$(\exists x)\, Gx$	4 EG

The restriction on EG prevents (3) from being shown valid

$$\begin{array}{ll} & \underline{(\exists y)\, aFy} \\ \therefore & (\exists y)(\exists y)\, yFy \end{array} \tag{3}$$

If the restriction "*ν does not occur in α*" were not there, then the conclusion of (3) would follow from the premise directly. But there is the following counterexample to (3):

F	a	b
a	0	1
b	0	0

By the elimination of quantifiers in (3) we get:

$$aFa \vee aFb \quad = \quad 0 \vee 1 \quad = 1$$
$$aFa \vee bFb \quad = \quad 0 \vee 0 \quad = 0$$

Note that the left-most quantifier in the conclusion is vacuous. So our rule allows the addition of vacuous quantifiers.

It should also be noted that formulae with free occurrences of individual variables should be Universally Closed before dealing with them in any proof.

EXERCISE 15.2

1. Construct deductions to show that the following are valid.

(a) $(\forall x)\, Fx \;/\; \therefore\; (\exists x)\, Fx$
(b) $Fa, (\forall x)(Fx \supset Gx) \;/\; \therefore\; (\exists x)(Gx \;\&\; Fx)$
(c) $(\forall x)(Fx \supset Gx),\; Ha \;\&\; Fa \;/\; \therefore\; (\exists x)(Hx \;\&\; Gx)$
(d) $\sim(\exists x)(Fx \;\&\; Gx),\; Ha \;\&\; Fa \;/\; \therefore\; (\exists x)(Hx \;\&\; \sim Gx)$
(e) $(\forall x)(Fx \supset Gx),\; Ha \;\&\; \sim Ga \;/\; \therefore\; (\exists x)(Hx \;\&\; \sim Fx)$
(f) $(\forall x)(\forall y)\, xRy \;/\; \therefore\; (\exists x)(\exists y)\, xRy$
(g) $(\forall x)(\forall y)\, xRy \;/\; \therefore\; (\exists x)\, xRx$
(h) $(\forall x)(\forall y)\, xRy \;/\; \therefore\; (\exists x)(\forall y)\, xRy$
(i) $(\forall x)(Fx \supset Gx) \vee \sim(\exists x)(Kx \;\&\; Lx),\; Fa \;\&\; \sim Ga,\; Kb \;/\; \therefore\; \sim Lb$
*(j) $(\forall x)(p \supset Fx) \;/\; \therefore\; p \supset (\exists x)\, Fx$

15.3 UNIVERSAL GENERALIZATION

The rule of Universal Generalization, UG), is complex because of conditions which govern its use. The rule is as follows.

UG α

$\therefore (\forall \nu)\, \alpha\, (\nu \,//\, \kappa)$ where κ is an individual constant.

provided that (i) κ does not occur in any premise or undischarged assumption.

(ii) κ does not occur in the dictionary.

This rule can be demonstrated only in the context of a deduction. We can use it to show (1) to be valid.

$$\begin{array}{ll} & (\forall x)(Fx \supset Gx) \\ & (\forall x)(Gx \supset Hx) \\ \hline \therefore & (\forall x)(Fx \supset Hx) \end{array} \qquad (1)$$

1.	$(\forall x)(Fx \supset Gx)$	P
2.	$(\forall x)(Gx \supset Hx)$	P / $\therefore$ $(\forall x)(Fx \supset Hx)$
3.	$Fa \supset Ga$	1 UI
4.	$Ga \supset Ha$	2 UI
5.	$Fa \supset Ha$	3, 4 Ch Ar
6.	$(\forall x)(Fx \supset Hx)$	5 UG

We can replace a in 5 by x because a does not occur in any premise or undischarged assumption, and there is no dictionary for (1).

The idea behind UG is that if the individual constant is utterly arbitrary, then everything has its properties and relations. In the deduction above a is utterly arbitrary.

If the proviso is ignored we can see what happens.

Let Fx = x is a farmer *Universe* = people
a = Alan

1.	Fa	P / ∴ $(\forall x)\, Fx$
2.	$(\forall x)\, Fx$	1 UG (ignoring the proviso)

The obvious counterexample is

	F
a	1
b	0

In the next section we introduce the use of Existential Instantiation (EI). Great care must be taken when both UG and EI are used in a deduction.

EXERCISE 15.3

1. In which of the following is UG correctly used, and in which is it not? If not, why not?

(a)

1.	$(\forall x)(Fa \supset Gx)$	P
2.	$Fa \supset Gb$	1 UI
→3.	Fa	A
4.	Gb	2, 3 AA
5.	$(\forall x)\, Gx$	4 UI
6.	$Fa \supset (\forall x)\, Gx$	3-5 CP

(b)

1.	$(\forall x)\, xRx$	P
2.	aRa	1 UI
3.	$(\forall y)\, aRy$	2 UG
4.	$(\exists x)(\forall y)\, xRy$	3 EG

(c)

1.	$Fa \supset (\forall x)\, Gx$	P
→2.	Fa	A
3.	$(\forall x)\, Gx$	1, 2 AA
4.	Ga	3 UI
5.	$Fa \supset Ga$	2-4 CP
6.	$(\forall x)(Fx \supset Gx)$	5 UG

(d)

1.	$(\forall x)\, Gx \supset Fa$	P
→2.	$\sim Fa$	A
3.	$\sim(\forall x)\, Gx$	1, 2 DC
4.	$(\forall x) \sim Fx$	2 UG
5.	$\sim Fa \supset (\forall x) \sim Fx$	2-4 CP
6.	$\sim(\forall x) \sim Fx \supset Fa$	5 Contra
7.	$(\exists x) \sim\sim Fx \supset Fa$	6 QN
8.	$(\exists x)\, Fx \supset Fa$	7 DN

(e)

1.	$(\exists x)\, Fx \supset Fa$	P
→2.	$(\forall x)\, Fx$	A
3.	Fa	2 UI
4.	$(\exists x)\, Fx$	3 EG
5.	$(\forall x)\, Fx \supset (\exists x)\, Fx$	2-4 CP
6.	$(\forall x)\, Fx \supset Fa$	1, 5 ChAr
7.	$(\forall y)((\forall x)\, Fx \supset Fy)$	6 UG

2. Construct deductions to show that the following are valid.

(a) $(\forall x)(Fx \supset Gx)$, $\sim(\exists x)(Hx \,\&\, Gx)$ / ∴ $(\forall x)(Hx \supset \sim Fx)$
(b) $(\forall x)(Fx \supset Gx)$ / ∴ $(\forall x)((Fx \,\&\, Kx) \supset Gx)$
(c) $(\forall x)(\forall y)\, xRy$ / ∴ $(\forall y)(\forall x)\, xRy$

(d) $p \supset (\forall x)\, Fx \;\; / \;\; \therefore\; (\forall x)(p \supset Fx)$

(e) $(\forall x)(p \supset Fx) \;\; / \;\; \therefore\; p \supset (\forall x)\, Fx$

*(f) $(\exists x)\, Fx \supset p \;\; / \;\; \therefore\; (\forall x)(Fx \supset p)$

(g) $(\forall x)(\forall y)(\forall z)((xRy \,\&\, yRz) \supset xRy), (\forall x) \sim xRx \;\; / \;\; \therefore (\forall x)(\forall y)(xRy \supset \sim yRx)$

(h) $(\forall x)(\forall y)(xRy \supset \sim yRx) \;\; / \;\; \therefore\; (\forall x) \sim xRx$

(i) $(\forall x)\, xRx \;\; / \;\; \therefore\; (\forall x)(\forall y)((xRy \vee yRx) \supset xRx)$

(j) $(\forall x)(Fx \,\&\, Gx) \;\; / \;\; \therefore\; (\forall x)\, Fx \,\&\, (\forall x)\, Gx$

15.4 EXISTENTIAL INSTANTIATION

The rule of Existential Instantiation, (EI), for natural deduction is quite different to the EI rule in truth-trees. EI in natural deduction is the most complex of the rules. We set the rule out:

EI $(\exists \nu)\, \varphi\nu$

$\varphi\kappa$

$\vdots$

α

α

where $\varphi\kappa$ is an itemization of $\varphi\nu$ to κ, and

(i) κ does not occur in any premise or undischarged assumption

(ii) κ does not occur in $(\exists \nu)\, \varphi\nu$

(iii) κ does not occur in α

We can use EI to show (1) valid.

$$\begin{array}{c} (\forall x)(Gx \supset Fa) \\ (\exists x)\, Gx \\ \hline \therefore\;\; Fa \end{array}$$

1.	$(\forall x)(Gx \supset Fa)$	P
2.	$(\exists x)\, Gx$	P / ∴ Fa
3.	Gb	A
4.	$Gb \supset Fa$	1 UI
5.	Fa	3, 4 AA
6.	Fa	2, 3-5 EI

Note how the justification is entered. Line 3 is gained from 2 by a move like the truth-tree EI. Line 5 does not contain b, so we can discharge the assumption to get Fa.

Consider some of the invalid moves which can occur when the proviso clauses are ignored for EI. Consider a deduction with the following dictionary in mind:

Universe = persons

b = Bill

c = Carol

Rx = x is rich

xLy = x loves y

1.	Rb	P
2.	$(\exists x)\, cLx$	P / ∴ $(\exists x)(cLx \,\&\, Rx)$
3.	cLb	A (ignoring proviso (i))
4.	$cLb \,\&\, Rb$	1, 3 Conj
5.	$(\exists x)(cLx \,\&\, Rx)$	4 EG
6.	$(\exists x)(cLx \,\&\, Rx)$	2, 3-5 EI

There is an obvious counterexample.

	R
b	1
c	0
d	0

L	*b*	*c*	*d*
b	0	0	0
c	0	0	1
d	0	0	0

Bill is rich. Carol is not rich, neither is *d*. Carol does not love Bill, but *d*. Carol loves someone, but not someone rich.

With the same dictionary we see what happens by ignoring proviso (ii)

1.	$(\forall y)(\exists x)\ yLx$	P / ∴ $(\exists x)\ xLx$
2.	$(\exists x)\ cLx$	1 UI
3.	cLc	A (ignoring (ii))
4.	$(\exists x)\ xLx$	3 EG
5.	$(\exists x)\ xLx$	2, 3-4 EI

Clearly this is invalid. We have the counterexample:

L	*a*	*c*
a	0	1
c	1	0

Everyone loves someone, but it is false that someone loves themselves

Consider what happens when proviso (iii) is ignored.

1.	$(\exists x)\ xLc$	P / ∴ bLc
2.	bLc	A
3.	bLc	A
4.	$bLc \supset bLc$	3-3 CP
5.	bLc	2, 4 AA
6.	bLc	1, 2-5 EI (ignoring (iii))

The counterexample is clear:

L	*b*	*c*
b	1	0
c	0	1

Someone loves Carol, but it is not Bill

EXERCISE 15.4

1. In which of the following is EI correctly used, and in which is it not? If not, why not?

(a)

1.	$(\forall x)(Fx \supset Gx)$	P
2.	$(\exists x)(Hx\ \&\ Fx)$	P
3.	$Ha\ \&\ Fa$	A
4.	Fa	3 Simp
5.	Ha	3 Simp
6.	$Fa \supset Ga$	1 UI
7.	Ga	4,6 AA
8.	$Ha\ \&\ Ga$	5,7 Conj.
9.	$(\exists x)(Hx\ \&\ Gx)$	8 EG
10.	$(\exists x)(Hx\ \&\ Gx)$	2,3-9 EI

(b)

1.	$(\exists x)(\exists y)\ xLy$	P
2.	$(\exists y)\ aLy$	A
3.	aLa	A
4.	$(\exists x)\ xLx$	3EG
5.	$(\exists x)\ xLx$	2, 3-4 EI
6.	$(\exists x)\ xLx$	1, 2-5 EI

(c)

1.	$(\exists x)(Fx \supset Ga)$	P
→ 2.	$Fb \supset Ga$	A
→ 3.	Fb	A
4.	Ga	2,3 AA
5.	$(\exists x)\, Gx$	4 EG
6.	$Fb \supset (\exists x)\, Gx$	3-5 CP
7.	$Fb \supset (\exists x)\, Gx$	1,2-6 EI
8.	$(\exists y)(Fy \supset (\exists x)\, Gx)$	7EG

(d)

1.	$(\exists x)(Fx \vee Gx)$	P
→ 2.	$Fa \vee Ga$	A
→ 3.	Fa	A
4.	$(\exists x)\, Fx$	3 EG
5.	$Fa \supset (\exists x)\, Fx$	3-4 CP
→ 6.	Ga	A
7.	$(\exists x)\, Gx$	6 EG
8.	$Ga \supset (\exists x)\, Gx$	6-7 CP
9.	$(\exists x)\, Fx \vee (\exists x)\, Gx$	2,3,8 CD
10.	$(\exists x)\, Fx \vee (\exists x)\, Gx$	1,2-9 EI

(e)

1.	$(\exists x)(Fx \supset Ga)$	P
→ 2.	$Fa \supset Ga$	A
→ 3.	$(\forall x)\, Fx$	A
4.	Fa	3 UI
5.	Ga	2,4 AA
6.	$(\exists x)\, Gx$	5 EG
7.	$(\forall x)Fx \supset (\exists x)Gx$	3-6 CP
8.	$(\forall x)Fx \supset (\exists x)Gx$	1,2-7 EI

2. Construct deductions to show that the following are valid.
   (a) $(\exists x)(Fx \supset p)\ /\ \therefore\ (\forall x)\, Fx \supset p$
   (b) $(\forall x)(Fx \supset Gx), (\exists x) \sim Gx\ /\ \therefore\ \sim(\forall x)\, Fx$
   (c) $(\exists x)(\exists y)\, xRy\ /\ \therefore\ (\exists y)(\exists x)\, xRy$
   (d) $(\exists x)(\forall y)\, xRy\ /\ \therefore\ (\forall y)(\exists x)\, xRy$
   (e) $(\forall x)(\forall y)\, xRy\ /\ \therefore\ (\forall x)\, xRx$

3. Prove the following theorems (ZPCs):
   (a) $(\forall x)(Fx \supset Gx) \supset ((\forall x)Fx \supset (\forall x)\, Gx)$
   (b) $(\exists x)\, Fx \vee (\exists x)\, Gx\ .\!\supset\!.\ (\exists x)(Fx \vee Gx)$
   (c) $(\exists x)(Fx \,\&\, Ga)\ .\!\equiv\!.\ (\exists x)\, Fx \,\&\, Ga$
   (d) $(\forall x)(Fx \,\&\, Ga)\ .\!\equiv\!.\ (\forall x)\, Fx \,\&\, Ga$
   (e) $(\forall x)\, Fx \equiv (\forall y)\, Fy$

Puzzle 15

A convoy of four trucks is on the road. The trucks are from an arms factory, a food store, a fuel store, and a building materials store. The drivers are an army sergeant, an RAAF sergeant, an army corporal and an army private. The army corporal is driving the first truck. The trucks each have a final destination. Those destinations are an Army base, an RAAF base, a Navy base and a civilian airfield.

The convoy commander has left his clip board of details behind. He receives a message that one of the trucks has a bomb in it, and that if the truck stops moving the bomb will explode. He also is told that the bomb is in the truck driven by a sergeant who is going to a destination not of his own branch of the armed forces. The commander jots down what he knows. It is as follows:

(i) The truck going to the Army base is ahead of the one going to the Navy base.

(ii) The truck from the arms factory is ahead of the one from the fuel store.

(iii) The truck from the building store is ahead of the one driven by the RAAF sergeant.

(iv) The truck going to the civilian airfield carries food.

(v) The truck driven by the army private is to the rear of the truck going to the Navy base.

(vi) The army sergeant drives a truck to the rear of the truck going to the RAAF base.

(vii) The corporal is not driving the truck from the fuel store or the building materials store.

(viii) The army private is driving the truck which is to go to the civilian airfield.

(ix) The RAAF sergeant is not driving to the army base.

Which truck holds the bomb?

Show how you arrived at your answer – (you have 30 minutes)!

15.5 SUMMARY

The system of *natural deduction for QT* subsumes that for PC (see Ch. 8) and adds two substitution rules (QN) and four inference rules (UI, EG, UG, EI). In specifying these rules we use the following notations: κ denotes any individual constant, $\alpha(\nu//\kappa)$ denotes the result of substituting ν for each occurrence of κ in α, and $\phi\kappa$ denotes an itemization of $\phi\,\nu$ to κ.

Quantifier Negation (QN):

$$\sim(\forall\nu)\alpha \;::\; (\,(\exists\nu)\sim\alpha$$
$$\sim(\exists\nu)\alpha \;::\; (\forall\nu)\sim\alpha$$

Universal Instantiation (UI):

$$\begin{array}{ll} & (\forall\nu)\,\phi\nu \\ \therefore & \phi\kappa \end{array}$$

(for *any* κ)

Existential Generalization (EG):

$$\begin{array}{l} \alpha \\ \therefore\ (\exists\nu)\ \alpha\ (\nu//\kappa) \end{array}$$

(where κ is not in α)

Universal Generalization (UG):

$$\begin{array}{l} \alpha \\ \therefore\ (\forall\nu)\ \alpha\ (\nu//\kappa) \end{array}$$

(where κ is not in any premise or undischarged assumption or the dictionary)

Existential Instantiation (EI):

$$\begin{array}{l} (\exists\nu)\phi\nu \\ \rightarrow\ \phi\kappa \\ \quad\vdots \\ \quad\alpha \\ \hline \quad\alpha \end{array}$$

(where κ is not in any premise or undischarged assumption, in $(\exists\nu)\,\phi\nu$ or in α)

Note that this version of EI is quite different from the EI used in trees.

POSTSCRIPT

Although we are nearly at the end of this text, we are nowhere near the end of logic. There are vast areas as yet to be explored. Some can be discovered by following propositional logic and quantificational logic into their many extensions. Some areas of logic are quite different to anything we have met in this text.

The extensions of propositional logic cover a wide range of modal and relevant logics. In the modal logics there are systems which attempt to formalize the notions of possibility and necessity. Other modal logics attempt the formalization of the notions of time, knowledge, obligation, and processes. Relevant logics attempt to give better formalizations of conditionals than is possible in ordinary propositional logic.

The extensions of quantificational logic include all the extensions of propositional logic, and more. There are four very important extensions to quantificational logic. They are the logic of *identity*, the theories of *definite descriptions*, the theory of *dyadic predicates*, and *set theory*. We have already met some set theory in chapters 9 and 13, but there is a more adequate and extensive basis for set theory in the extensions of QT. It is in this area that we find the logical tools to discuss the foundations of mathematics and of logic itself.

Most of the logic we have considered descends from the work of Gottlob Frege. But not all logic in the modern world is so heavily in debt to Frege. One other important sort of logic is mereology, a logic of parts and wholes, developed by the Polish logicians, in particular Leśniewski.

We hope that you will pursue logic further. There are many fascinating things in the far reaches of the abstract, and many amusing things in the logical puzzles and paradoxes of Lewis Carroll, or Raymond Smullyan and others. So, in conclusion we hope that it is not the conclusion.

APPENDIX 1
TRADITIONAL LOGIC

Traditional Logic derives mainly from the system which Aristotle (384-322 B.C.) set out in the *Prior Analytics*. Although the system is very limited in scope, it is important in two ways. First, it is the first system of formal logic in the history of logic. Second, it incorporates, in one version, an unusual logical feature which is worth noting.

Traditional logic deals only with arguments in which all propositions are of either **A, E, I** or **O** form. The logical form of these is set out in ordinary language, e.g:

A : *Every S is P*　　E : *No S is P*
I : *Some S is P*　　O : *Some S is not P*

Propositions of these forms are usually called *categorical* propositions. The **A** and **E** are of universal ***quantity***, **the I and O** of particular ***quantity***. The **A** and **I** are of affirmative ***quality*** (*a*ff*i*rmo), the **E** and **O** of negative ***quality*** (n*e*g*o*).

The letters *S* and *P* are *term variables.* Any capital letter may be used. In each of the above *S* is said to be in *subject* position, and *P* in *predicate* position. So, in "Every frog is green", the *subject term* is "frog" and the *predicate term* is "green".

Arguments which satisfy the following conditions are *syllogisms*:

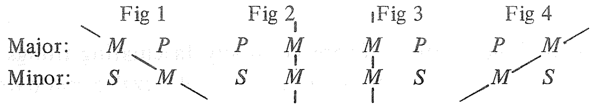
(a) there are two premises,
(b) the subject term of the conclusion occurs in one premise, and
the predicate term of the conclusion occurs in the other premise,
(c) there is one term in common between the premises.

In a syllogism the predicate term of the conclusion is known as the *major term*, and the premise in which it occurs is the *major premise*. The subject term of the conclusion is the *minor term*, and the premise in which it occurs is the *minor premise*. When set out in standard form a syllogism has the major premise as the first premise, e.g:

Major Premise : No dog is a cat
Minor Premise : Every kelpie is a dog
Conclusion : ∴ No kelpie is a cat (1)

The major term is "cat", the minor term is "kelpie". The third bridging term, is known as the *middle term*. Here it is "dog".

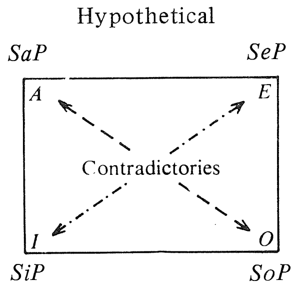
The logical form of any syllogism can be described in terms of its *mood* and *figure*. The *mood* sets out, in order, the form of the premises and conclusion. In (1), the mood is EAE. The *figure* indicates the arrangement of terms in the premises. There are four possible arrangements, depending on the alignment of the middle terms.

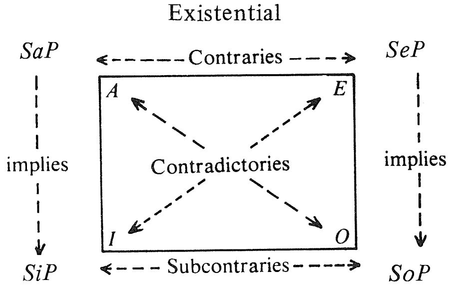


The lines act as a useful reminder. Argument (1) is in Figure 1.

Syllogisms may be assessed for validity from either the *Hypothetical Viewpoint* or the *Existential Viewpoint*. When syllogisms are assessed from the Hypothetical Viewpoint the A, E, I and O propositions are treated as they would be in MQT. The square of opposition is as below. Contrast the two squares.

We will use *SaP, SeP, SiP* and *SoP* for the **A E I** and **O** respectively:





The **A** and **E** are not contrary from the Hypothetical Viewpoint because there might be no *S* i.e. *S* could be an *empty term.* Also, if *S* is empty, the **A** can be true and the **I** false, so **A** does not imply **I**. In like fashion, from the Hypothetical Viewpoint the **E** does not imply the **O**, nor are the **I** and **O** sub-contrary.

In the Existential Viewpoint we assume that no subject term is empty. When using diagrams to assess for validity from the Existential Viewpoint, care must be taken to ensure that every subject term occurring in the argument is non-empty. If you think of this in terms of MQT, the Existential Viewpoint is like adding to every syllogism as extra premises (2) and (3) or (4) or both, depending on the figure:

$$(\exists x)\, Sx \tag{2}$$
$$(\exists x)\, Mx \tag{3}$$
$$(\exists x)\, Px \tag{4}$$

There are fifteen valid syllogisms, described in terms of mood and figure, from the Hypothetical Viewpoint. There are an extra seven when the Existential Viewpoint is taken. The additional seven do not each require the existential viewpoint for all the subject terms. So we set out a table of valid forms below, with the presuppositions necessary:

	Fig 1	Fig 2	Fig 3	Fig 4	Presupposition
Hyp	AAA EAE AII EIO	EAE AEE AOO EIO	IAI AII OAO EIO	EIO AEE IAI	Nil
Exis	AAI EAO	AEO EAO		AEO	$(\exists x)\, Sx$
			AAI EAO	EAO	$(\exists x)\, Mx$
				AAI	$(\exists x) Px$

Traditionally, it was claimed that the subject terms of universals, **A** and **E**, and the predicate terms of negatives, **E** and **O**, were ***distributed***. The other terms were ***undistributed***. The general idea was that in a proposition, if the assertion is about all things to which the term refers then that term in that position in that proposition is distributed. Whatever the rationale for distribution, if we treat it formally as above, the notion can be used with the following five rules to give a short cut assessment method for syllogisms from both Viewpoints.

1. In any valid syllogism the middle term is distributed at least once.
2. In any valid syllogism no term is distributed in the conclusion unless it is distributed in a premise.
3. No valid syllogism has two negative premises
4. Any valid syllogism has one negative premise if and only if it has a negative conclusion.
5. No syllogism valid from the Hypothetical Viewpoint has two universal premises and a particular conclusion.

The syllogistic technique may be extended to handle certain longer arguments which have more than two premises. Consider for instance the argument

All logicians are nice people
Some students are logicians.
All students are seekers of knowledge.

∴ Some seekers of knowledge are nice people.

We cannot evaluate this as a single syllogism since there are more than two premises. However the first two premises may be used to yield the syllogistic conclusion "Some students are nice people" which, when combined with the third premise, syllogistically implies the final conclusion. The argument may thus be treated as a *chain of syllogisms* wherein the intermediate conclusions have to be supplied. In the example above there were two links in the chain, but in practice the chain may be as long as we wish. Any such argument with three or more premises is known as a *sorites* (from the Greek word for a pile). Lewis Carroll, in his logic books, has many delightful sorites, some with as many as fifty premises.

Arguments in which there is one premise only, and in which the conclusion terms are the same or the negatives of the premise terms, are called *immediate inferences*. If we allow * to be any one of **a e i** or **o**, then the following syntactical terminology is used:

the *converse* of **A** * **B** is **B** * **A**
the *contrapositive* of **A** * **B** is non-**B** * non-**A**

Any immediate inference in which the conclusion is the converse of the premise is known as *conversion*. Similarly, when the conclusion is the contrapositive of the premise the inference is known as *contraposition*. Clearly, conversion with either an **I** or **E** premise is valid, and contraposition with either an **A** or **O** premise is valid.

The inference *SaP* / ∴ *PiS* is known as *conversion per accidens*, and is valid from the Existential Viewpoint.

The *obverse* of a categorical proposition is that proposition which has the same terms in the same position, but the predicate is negated and the quality, but not quantity, is changed. In the following pairs the right is the obverse of the left:

SaP – *Se non-P*   *SeP* – *Sa non-P*
SiP – *So non-P*   *SoP* – *Si non-P*

Since any categorical proposition is logically equivalent to its obverse, *obversion* is a valid inference.

APPENDIX 2
CRITICISMS OF DEDUCTION

From at least Plato's time, various attempts have been made to depreciate the value of deduction. The most well known critic in this regard was John Stuart Mill (1806-1873), who in his *System of Logic* argued that deduction is never the source of new knowledge, the sole method of making real inferences being induction. Let us now consider briefly two main criticisms: deductive reasoning begs the question; deduction is merely a disguised form of induction.

The argument that deduction is question-begging runs something like this. To be of any use a deductive argument must be valid. But in a valid argument the conclusion can contain no information not already in the premises. Hence such an argument is useless and question-begging since its conclusion must already be assumed in the premises. Consder the following two examples.

All footballers on our team are right handed.
Bozo is a footballer on our team.

∴ Bozo is right handed . . .(1)

All bachelors are male.
Tom is a bachelor.

∴ Tom is male. . . .(2)

In order to establish the first premise of argument (1) we would have to already know the conclusion. In argument (2) we cannot know the second premise without first knowing the conclusion.

The above criticism may be countered fairly readily. Even with the two favourable arguments cited it must be admitted that they would not be question-begging for *some* people e.g., in argument (1) a person might learn the first premise simply by hearing it from a reliable source. This is only a minor objection however. A firm rebuttal of the criticism is achieved by giving other examples of deduction which are plainly not question-begging. Fortunately such examples abound in the logic we have already studied. For instance, in relation to the detective's argument in §8.1, the conclusion that the burglar came through the door is not something that must be known before the premises can be known. Anyone who has sweated over a difficult problem in logic, mathematics or even domestic matters has a right to feel elated when he deduces the answer: by his reasoning he has proved something new to himself. If we look at even such a simple case as Disjunctive Addition ($p \ / \ \therefore p \vee q$), it is clear that when it is said that the conclusion of a valid argument is "contained" in the premises, this containment is one of implication rather than obvious display. Often, the conclusion is so deeply hidden in the premises that the discovery of its truth can be both difficult and novel. Once we go beyond the traditional syllogism (the focus of most critics of deduction) it is easy to find examples of deduction which are not question-begging.

Let us now consider the second criticism. In response to the view that deduction helps us make explicit what was only implicit in the premises, Mill demanded an explanation for how something could be implicitly contained in what we already know. Consider the following argument.

All men are mortal.
Tom is a man.

∴ Tom is mortal. ...(3)

Mill expounded a theory wherein a generalisation like "All men are mortal" is no more than a mental note or register of our observations that many particular men have died: when we "deduce" that some other living person is mortal this is actually in inductive inference from our previous observations of people who died. The inductive inference to the generalisation "All men are mortal" was useful, but the deduction from this generalisation to a particular instance provides no new information and is only an "apparent inference". Thus, according to Mill, only in induction do we make any "real inferences". When applied to well chosen examples like argument (3), Mill's theory seems plausible enough. But to identify, say, the derivation of Green's Theorem from the axioms of vector analysis, as an exercise in inductive analogy, is clearly ludicrous. Mill did attempt to overcome such a counter move by developing a theory whereby the axioms of mathematics are derived inductively from our experiences. This psychological theory was never accepted by other philosophers, and is incompatible with the modern approach to axiomatic theory which regards axioms as arbitrary assumptions rather than "obvious truths". The formal treatment of deductive systems has zero inductive content, and the testing for isomorphism between the formal theory and physical reality is purely deductive: if a prediction from the theory fails, the isomorphism does too; if not the theory may be tentatively (but never certainly) accepted until a later prediction fails. Karl Popper and others have extended this hypothetico-deductive viewpoint to not only rebut but reverse the position of Mill, reducing induction to a species of deduction (hypothetico-deduction).

Answers

Ex. 1.2

1. a,c,d,g,h,i,j,l,m,p,q (only if "Super-sausage" refers), r,s,u,w.
2. (a) John himself made the mince. John made the mince out of his own hands. (b) The lamb feels so hot it does not want to eat. The temperature of the piece of cooked lamb is too high for people to eat it. The lamb was recently stolen and should not be eaten. (c) To visit relatives can be a nuisance. Relatives who visit can be a nuisance. (d) Students dislike lecturers who are boring. Students don't like to bore lecturers. (e) Some dogs have no sense of smell. Some dogs have no odour. (f) He is speaking about old languages. He is speaking about old animal tongues. (g) The game of cricket stopped when the cricket bat began to squeak. The cricket stopped when the other animal, a bat, began to squeak. (h) Sons but not daughters are spoilt. Sons with no brothers are spoilt. (i) He spoke to the chairman. He wrote an address on the chair while on the floor. (j) The wind caused the person bowling to lose his temper. The bowler hat was blown off the handle by the wind. (k) Tom, the American Indian, would make an effort. Tom would judge the indian.
3. (a) Condensed short courses, or courses on how to crash. (b) "may" for "may possibly" or "are allowed to". (c) Police record, or musical record. (d) ambiguous left-scope of "–free" (e.g., was the diet meat-free?) and ambiguous right-scopes of "honey-dipped" and "curried" (e.g. was the rice curried?).
4. Reading "same" roughly as "equivalent" gives: a, c, f, g, h, i, j. Some of these are debatable with a stronger reading of "same" e.g., can "north" and "south" be conceptually distinct?
5. c, e, g, i (debatable).

Puzzle 1.

One analysis is as follows. Since Epimenides was a Cretan, what he said, if true, would be a lie and hence both true and false. So what he said cannot be true. If some Cretans tell the truth sometimes, then what Epimenides said is simply false. So we may conclude that Epimenides did utter a proposition (a false one). Now try your hand at the following sentence, known as the *pseudomenon* or the *liar paradox*: "What I am now saying is false.". A standard analysis of this is that it does not express a proposition because the utterance is true if and only if it is false (Why?). For further discussion of logical paradoxes see W. V. Quine's "Paradox", *Scientific American* (1962, April) and for some nice points about self-reference see D. R. Hofstadter's "Metamagical Themas", *Scientific American* (1981, Jan).

Ex. 1.3

1. Contradictories: a, c, e, g, i, Contraries: b (assuming men exist), d, f, h, j, k.
2. (a) John is not sick. (b) Jack is not Bill's brother. (c) Jack and Jill are not both hill climbers. (d) Wales is not smaller than Queensland. (e) Jack is not Australian or Jill is not Scottish. (f) Sometimes it rains. (g) Not all men are mortal. (h) It's impossible that you left it in the rain. (i) Some fools are rich. (j) No students are very wise.
3. Assuming the subject term exists, and writing the negation first: (a) That number is not positive. That number is zero. (b) Paul did not come first in the race. Paul came third in the race. (c) My favourite recording artist is not Donovan. My favourite recording artist is Neil Diamond. (d) He is not 33 years old. He is 35 years old. (e) The universe did not begin with a big explosion 16 billion years ago. The universe has always existed. (f) The colour of the car is not red. The colour of the car is blue.
4 (a) If a contrary of *p* is true then *p* can't be true. So *Not p* is true, and this is a contradictory of *p*. All other contradictories of *p* are equivalent to *Not p*. (b) *p* is false, but this reveals nothing about its contraries.
5. (a) No. (b) Yes. (c) Logic is not very interesting.
6. They could both be false e.g., Tom may not be enrolled in the subject.
7. a, b. c is false e.g., *Not p* is the negation of *p*, but not conversely.
8. (a) Here "in" meant "into" rather than "not", so "inflammable" meant "highly flammable"; some people however read it as "non-flammable". (b) "incorrect". "infamous" means "notorious" and "invaluable" means "priceless".

Ex. 1.4

1. (a) The workmen put down their tools. Brown made a speech. (b) Michael is slow. Michael is careful. (c) Alan is here. Betty is here. Colin is here. (d) The gates are not locked. Neither the side door nor the back door is closed. (e) The burglar is not in the house. The burglar will be either on the road or on the moors. (f) If anyone is sick they should see the doctor. It is clear that Bill is not well. (g) If the bus has gone then my watch is slow. If my watch is slow then the tower clock is slow.
2. a, d (b: Jane is Mary's sister. c: share the same room. e: went up together). b is allowed if "sister" means "religious sister" or the like.
3. (a) James went to the library. James went to the club. Inclusive. (b) Mary is to enroll in maths. Mary is to enroll in physics. Exclusive. (c) He studied French. He studied logic. In. (d) The number is less than 10. The number is greater than 20. Ex. (e) The person who chose that colour scheme was colour-blind. The person who chose that colour scheme was lacking in good taste. In. (f) The rain will come and the crop will be planted. We will sell the farm. Ex. (g) The number is not more than 10. The number is greater than 6. In. (h) Mary takes maths and logic. Mary takes Japanese and computing. Ex.
4. Contradictories: a, b, d, e. Contraries: c.
5. (a) Susan is neither a clerk nor a teacher. (b) Sandy is not both a farmer and an accountant. (c) It's not that both the bus is slow and time is running out. (d) Neither is the bus slow nor am I impatient (e) Not both Robin and Chris are mechanics. (f) Cathy is either beautiful or not attractive . (g) You will not finish your homework before 9.30 and you will watch T.V. after 9.30.

Ex. 1.5

1. (a) Taxes are cut. People will spend more money. (b) Snoopy is a dog. Snoopy is an animal. (c) Tom believes he is being helped. Tom is acting in a strange way (d) Fuzzy is a bear. Fuzzy is hairy. (e) Fuzzy is a bear. Fuzzy is an animal. (f) Neither Brown nor Jones breaks the law. Brown and Jones have nothing to fear. (g) The wheat will grow. The wheat is planted. (h) It rains. Either there will be a flood or the crops will be ruined. (i) Conditions are not completely sterile. The experiment will not be successful.
2. a, c, d.
3. (a) If people ... money then taxes ... cut. (b) If Snoopy ... animal then Snoopy ... dog. (c) If Tom ... way then he believes ... helped. (d) Fuzzy ... hairy only if she ... bear. (e) Fuzzy ... bear if Fuzzy ... animal. (f) If Brown ... fear then neither ... law. (g) The wheat ... planted only if it ... grow. (h) If either ... spoiled then it rains. (i) If the ... successful then conditions ... sterile.
4. (a) If the number is even then it is divisible by two. If ... two then ... even.
   (b) If there will be an election then the Governor-General signs the writs. If ... writs then ... election. (c) If the experiment will be a success then the correct procedures are followed. If ... followed then ... success. (d) We will go on a picnic if it doesn't rain. We ... picnic only if ... rain.

Ex. 1.6

1. a, d, e, f, g.

Ex. 1.7

1. a, d; b, f; c, h; e, g.
2. (a) *If D then M, If H then M, D or H / ∴M* (b) *R only if M, M / ∴ R* (c) *If N then F, N / ∴ F* (d) *Not (P and C), P / ∴ Not C* (e) *If T then S, Not S / ∴ Not T* (f) *(G and H) or D, Not (G and H) / ∴ D* (g) *If M then S, Not M / ∴Not S* (h) *If T then N, if N then C, If C then E / ∴ If T then E* (i) *K only if F / ∴ If not F then not K* (j) *G only if R, R / ∴ G* (k) *If C then E, Not E / ∴ Not C* (l) *If D then A, Not D / ∴ Not A* (m) *If R then N, If N then L, If L then C / ∴ If R then C* (n) *Not (I and A), I / ∴ Not A* (o) *S only if U / ∴ If not U then not S* (p) *If R then W, If H then W, R or H / ∴ W* (q) *(R and F) or D, Not (R and F) / ∴ D* (r) *If V then L, V / ∴ L.*
   Matched pairs: a, p; b, j; c, r; d, n; e, k; f, q; g, l; h, m; i,o.

Ex 1.8

1. (a) A (b) C (c) B (d) C (e) D   2. a, c   3. a
4. (a) If Spinoza was a Queenslander then he was an Australian. Spinoza was a Queenslander / ∴ Spinoza was an Australian. VALID. (b) Hitler was a fascist. / ∴ Hitler was a fascist. VALID. (c) You can't be both a Christian and a Communist. You're not a Christian. / ∴ You're a Com-

munist. INVALID. (d) If the universe is part of God then God is imperfect. If the universe is not part of God then God is imperfect. The universe is either part or not part of God / ∴ God is imperfect. VALID. (e) Queensland is hot. The Northern Territory is hotter. / ∴ The Northern Territory is very hot. INVALID.

5. Code: T = True, F = False. (a) F e.g., Question 1c. (b) F e.g., 1c. (c) T. (d) T. (e) T. (f) T. (g) F e.g., Lemons are sweet fruit / ∴ Lemons are fruit. (h) F e.g., Lemons are fruit. / ∴Lemons are sour fruit. (i) F e.g., Lemons are fruit. / ∴ Lemons are sour. (j) T. (k) F e.g., 5h. (l) T. (m) T. (n) F e.g., 5h. (o) T. (p) F e.g., 4b (however the argument has no persuasive value: it is said to "argue in a circle"). (q) F e.g., remove the first premise from: I like birds. I like cats. All cats and birds are animals. / ∴ I like some animals.
6. All are valid except for Question 2 b, j, g, l.

Ex 2.2

1. c, f, j, k, l, o.
2. (a) B
   1 R$\sim$
   1, 2 R&
   3 R$\sim$

   (d) B
   B
   1, 2 R$\not\equiv$
   1 R$\sim$
   2 R$\sim$
   4, 5 R&
   1, 2 R&
   6, 7 R$\vee$
   3, 8 R$\not\equiv$

3. (e)

1.	p	B
2.	q	B
3.	$\sim q$	2, R$\sim$
4.	$(p \equiv \sim q)$	1, 3, R$\equiv$
5.	$\sim(p \equiv \sim q)$	4, R$\sim$
6.	$\sim p$	1, R$\sim$
7.	$(\sim p \equiv q)$	6, 2, R$\equiv$
8.	$\sim(\sim p \equiv q)$	7, R$\sim$
9.	$\sim\sim(\sim p \equiv q)$	8, R$\sim$
10.	$(\sim(p \equiv \sim q) \supset \sim\sim(\sim p \equiv q))$	5, 9, R$\supset$

4. (a) (i) Yes (ii) No (iii) Yes   5. (a) (i) No (ii) Yes (iii) Yes

Ex. 2.3

1. (a) $p \supset q \;.\equiv..\; \sim p \vee q$ (b) $p \,\&\, (q \vee p) \;.\supset.\; (q \,\&\, q) \supset p$ (c) $(p \supset q) \supset [(q \supset r) \,\&\, (r \supset s) \;.\supset.\; (p \supset s)]$
2. (a) $((p \,\&\, q) \equiv (q \,\&\, p))$ (b) $(((p \supset q) \,\&\, p) \supset q))$ (c) $\sim((p \,\&\, (q \vee r)) \not\equiv ((p \,\&\, q) \vee (p \,\&\, r)))$
3.

$\sim p$	$\sim q$	$q \supset p$	$p \supset p$
0	0	1	1
0	1	1	1
1	0	0	1
1	1	1	1

4.

&	1 0
1	1 0
0	0 0

$\vee$	1 0
1	1 1
0	1 0

$\equiv$	1 0
1	1 0
0	0 1

$\not\equiv$	1 0
1	0 1
0	1 0

5.

p	q	$p \subset q$
1	1	1
1	0	1
0	1	0
0	0	1

Ex 2.4A

1. (a) 1. (b) 4. (c) 2 (d) 3. (e) 1, 2, 3. (f) 2, 3, 4. (g) 1, 2, 4. (h) 1, 3, 4.
   (i) Ambiguous scope of "not": "(not tall) "gives 3; "not (tall and leafy)" gives 2, 3, 4.
   (j) 2, 3, 4. (k) 3. (l) Ambiguous between m and n. (m) 1, 3, 4. (n) 4. (o) 4. (p) 1, 3.
   (q) 1, 2 or 1, 4 or 2, 3 or 2, 4 or 3, 4. (r) 2, 4.
2. (a) The tree is tall. (b) The tree is not tall. (c) The tree is leafy and beautiful. (d) The tree is tall or leafy. (e) The tree is tall or leafy but not both. (f) The tree is leafy and beautiful but not tall. (g) The tree is not tall and not leafy. (h) The tree is not both tall and leafy. (i) The tree is neither tall nor leafy but it is beautiful.
3. (a) $P \,\&\, G$ (b) $\sim T$ (c) $G \vee B$ (d) $\sim\sim B$ (e) $(B \,\&\, T) \,\&\, G$ (f) $S \,\&\, T$ (g) $\sim S \,\&\, \sim T$ or $\sim(S \vee T)$ (h) $\sim(S \,\&\, T)$ (i) $\sim S \vee \sim T$ (j) $T \not\equiv S$

Ex 2.4B

1. (a) Today is Friday. (b) Today is not Friday. (c) Today is Friday and tomorrow is Saturday. (d) Today is Friday or tomorrow is Saturday. (e) If today is Friday then tomorrow is Saturday. (f) Today is Friday if and only if tomorrow is Saturday. (g) Today is either Wednesday or Friday, but not both.
2. a, c, d, e, f, g   3. b, e, f
4. (a) If I am a man or woman then I am human. (b) I am human if and only if I am a man or a woman. (c) I'm not both a man and a woman. (d) If I'm a man then I'm a human but not a woman. (e) If I'm human then I'm either a man or a woman but not both. (f) If I'm not a human

then I'm neither a man nor a woman. (g) If I'm a human then I'm not both a man and a woman.

5. (a) I (b) $\sim B$ (c) $I \vee U$ (d) $I \& U$ (e) $I \& U \& \sim B$ (f) $B \supset M$ (g) $M \supset N$ (h) $I \not\equiv B$ (i) $M \equiv N$ (j) $\sim B \& \sim N$ or $\sim(B \vee N)$ (k) $\sim(U \& B)$
6. (a) $\sim E$ (b) $E \vee \sim E$ (c) $O \supset \sim E$ (d) $\sim E \supset O$ (e) $O \equiv \sim E$ (f) $N \supset \sim P$ (g) $\sim(E \& O)$ (h) $\sim P \& \sim N$ *or* $\sim(P \vee N)$ (i) $Z \supset (\sim P \& \sim N)$ (j) $P \vee Z \vee N$ (k) $\sim Z \supset (P \not\equiv N)$ (l) $N \supset \sim P$ (m) $E \& \sim Z \& P$ (n) $E \equiv \sim O$ (o) $(E \vee P) \supset \sim(O \& N)$ (p) $(E \& P) \not\equiv (O \& N)$ (q) $P \supset (\sim Z \& \sim N)$

Ex 2.5

1. (a) $H \equiv J$ (b) $\sim W \& C$ (c) $(M \vee T) \supset \sim W$
2. (a) $p \equiv q; p$ (b) $\sim p \& q;\ p \& q;\ p$ (c) $(p \vee q) \supset \sim r;\ p \supset \sim q;\ p \supset q; p$
3. No. For example H and J each have the explicit form p.

Ex 2.6

1. (a) People understand how others feel if and only if they empathize with one another. (b) If either people are selfish or the inflation rate doesn't drop then people will be unhappy. (c) The inflation rate won't drop if and only if people are both selfish and ignorant of how others feel. (d) If people are selfish then neither will the inflation rate drop nor will they be happy. (e) The proposition that people will be happy if and only if both the inflation rate drops and they are both unselfish and understanding of how others feel, is false.
2. (a) $S \supset (\sim E \& \sim H)$ (b) $(D \& H) \vee S$ (c) $\sim((D \& S) \supset H)$ (d) $\sim S \supset (\sim E \& \sim U)$ (e) $\sim S \supset (E \& H)$
3. (a) $C \supset F$ (b) $\sim(C \& P)$ (c) $C \& \sim P$ (d) $F \vee (P \& C)$ (e) $C \supset (P \supset \sim F)$ (f) $(C \not\equiv P) \vee \sim F$ (g) $\sim F \supset (C \& P)$ (h) $F \supset \sim(C \& P)$ (i) $\sim P \supset C$ (j) $\sim(P \vee C) \supset F$ (k) $\sim F \equiv (C \& P)$ (l) $\sim C \vee \sim P \vee \sim F$ (m) $F \supset (\sim C \vee \sim P)$ (n) $(\sim F \supset C) \& \sim(C \supset \sim F)$
4. (a) If Clark Kent disappears and Superman appears then Lois Lane becomes suspicious.
   (b) If Superman appears then, Lois Lane does not become suspicious if and only if Clark Kent does not disappear. (c) $(A \& \sim D) \supset \sim I$ (d) $[(L \vee I) \supset (A \& D)] \& \sim[(A \& D) \supset (L \vee I)]$
5. (Dictionary not supplied here) (a) E (b) $H \& \sim M$ (c) $A \& B$ (d) $(T \& B) \supset S$ (e) $\sim S$ (f) $D \supset (I \& E)$ (g) $D \& (E \supset R)$ (h) $\sim(M \vee A)$ (i) $S \not\equiv N$ (j) $(M \supset H) \& \sim(H \supset M)$
   (k) $M \& B \& N$ (Actually this is better translated as the argument $M, N / \therefore B$) (l) $L \supset S$
   (m) $U \& \sim(N \vee P \vee D \vee L \vee S \vee I)$ (n) $\sim(I \equiv (N \& L \& S))$ (o) $(P \& D \& \sim S) \supset \sim(\sim N \supset (L \supset I))$
6. (a) $\sim(R \supset B) \& \sim(B \supset R)$ $\overset{\sim}{\text{(b)}}[(A \not\equiv B \not\equiv C) \& \sim(A \& B \& C)]$ c) $(A \& D) \& (M \vee L)$
   (d) $\sim[(R \& A) \supset B] \& (\sim R \supset \sim B)$ (e) $(W \& A) \supset (T \& H)$ *or* $W \supset [A \supset (T \& H)]$

Puzzle 2 For tying horses to.

Ex 3.2

1. (a) $\underset{2}{\sim}(p \underset{1}{\supset} p)$ (b) $p \underset{2}{\equiv} \underset{1}{\sim} q$ (c) $(p \underset{1}{\supset} q) \underset{4}{\not\equiv} (\underset{2}{\sim} q \underset{3}{\&} p)$ (d) $p \underset{1}{\vee} q \underset{2}{\vee} r$ (e) $(p \underset{1}{\&} q) \underset{4}{\vee} (r \underset{3}{\supset} \underset{2}{\sim} s)$
2. To conserve space, answers to truth table questions will normally consist only of the main column from top to bottom but laid out left to right. It is assumed that the matrix is in the standard order with PVs listed left to right in alphabetical order.
   (a) 00 (b) 0110 (c) 1111 (d) 11111110 (e) 1111011101110111
3. (a) 32 row matrix with $p = 1$ on first 16 rows. (b) 2^7 i.e. 128
4. (a) $\overset{3}{(}\overset{2}{(}\overset{1}{(}p \& q\overset{1}{)} \supset r\overset{2}{)} \supset p\overset{3}{)}$ (b) $\sim\overset{4}{(}\overset{2}{(}q \vee \overset{1}{(}p \equiv r\overset{1}{)}\overset{2}{)} \supset \sim\overset{3}{(}r \& q\overset{3}{)}\overset{4}{)}$

   (c) $\sim\overset{7}{(}\sim\overset{5}{(}\sim q \equiv \overset{4}{(}\overset{1}{(}s \vee p\overset{1}{)} \supset \sim\overset{3}{(}\overset{2}{(}p \& \sim q\overset{2}{)} \vee r\overset{3}{)}\overset{4}{)}\overset{5}{)} \supset \overset{6}{(}p \supset q\overset{6}{)}\overset{7}{)}$

Ex 3.3

To conserve space the following code will be used: T = tautology; F = Contradiction; C = contingency.

1. (a) F (b) C (c) C (d) C (e) C
2. (a) C (b) C (c) C (d) T (e) F (f) C (g) C (h) C (i) F (j) C (k) C (l) C (m) T (n) T (o) T (p) F (q) C (r) F (s) C (t) T
3. $\sim(q \vee p)$ implies $\sim p$. $p \& \sim p$ is a contradiction. The negation of a contradiction is a tautology. So the formula is a tautology.

Ex 3.4

1. Code: T = Necessary Truth; F = Contradiction; C = Contingency.
   (a) C (b) F (c) F (d) T (e) C (f) T (g) F (h) T (i) T (j) T (k) C (l) F (m) T (n) T (o) C (p) T (q) T
2. (a) F (b) T (c) C (d) F (e) T
3. (a) True (b) False (c) False (d) True (e) True
4. Code: LP = Logically possible; PP = Physically possible; A = Allowed
   (a) PP (b) PP (c) PP (d) PP (e) LP (f) Can PP; may A (g) Ambiguous: PP or A

Puzzle 3 The simple answer is that Dr. What lives at the North Pole. There are however infinitely many other places on this planet where Dr. What could live and still satisfy the conditions given. See if you can find these. (*Hint*: Draw a sphere and consider lines of latitude very close to the poles.)

Ex 3.5

1. Code: T = tautology; F = PC-contradiction; I = PC-indeterminacy.
   (a) I (b) T (c) F (d) T (e) I (f) I (g) T (h) I (i) T (j) T (k) I (l) F (m) T (n) T (o) T
2. (a) F (use A = John has a red car and a blue car). (b) (ii) F (c) f T
3. (a) T (b) None

Ex 3.6

Code: T = Necessary Truth; F = Contradiction; C = Contingency.

1. (a) T (b) C (c) C (d) F (e) C
2. (a) T (b) C (c) C (d) F (e) T (f) C (g) C (h) F (i) C (0 when $C = 0, P = 1, S = 1$)

Ex 3.7A

1. a, f; b, h; c, g; d, e. 2. a, d; b, c.
4. (b) every proposition (hence an infinite number). (c) same answer as for (b).

Ex 3.7B

1. (i) 2 (ii) 1 (iii) 3 (iv) 4 (v) 7 (vi) 5 (vii) 6 (viii) 3 (ix) 5 (x) 7 (xi) 4 (xii) 3 (xiii) 4 (xiv) 6 (xv) 5 (xvi) 6.
2. Code: T = True; F = False. (i) F (ii) T (iii) F (iv) F (v) T (vi) T (vii) F (viii) T (ix) T
3. (i) 6 (ii) 1 (iii) 2 (iv) 6 (v) 7 4. (i) 2 (ii) 5 (iii) 7 (iv) 7 (v) 4
5. Rows with $L=0, U=1$ are eliminated. (i) 5 (ii) 1 (iii) 2 (iv) 3 (v) 5 (vi) 7
6. Consistent: (i), (ii), (v), (vi). Inconsistent: (iii), (iv)
7. No rows eliminated. (a) contraries (b) contradictories (c) contraries
8. Rows with $D = 0, L = 1$ are eliminated. (a) No (b) Yes (c) $C=D=L=1$ (.e. a world in which Fred has a cat and a large dog); $C=D=1, L=0$ (Fred has a cat and a dog that is not large); $C=1, D=L=0$ (Fred has a cat but no dog); $C=0, D=1, L=0$ (Fred has a non-large dog and no cat). (d) $C=0, D=L=1$ (Fred has a large dog but no cat); $C=D=L=0$ (Fred has neither a cat nor a dog). (e) No (f) Yes.
9. (a) No (b) Yes (c) $C=D=1, L=0$ (Fred has a cat and a non-large dog) (e) Put these into words: $C=D=L=1$; $C=1, D=L=0$; $C=0, D=L=1$; $C=0, D=1, L=0$; $C=D=L=0$ (e) Yes (f) No.
10. True: a, b, d, g, h, j, k, o False: c, e, f, i, l (e.g., both contradictions), m (e.g., both tautologies), n (e.g., p a contradiction and q a tautology).
11. Treating "sure" as synonymous with "certain", it follows that the student was inconsistent. If he can't be sure of anything then he can't be sure that he can't be sure of anything. (As a related research topic, examine the philosophical view that in order for an assertion to be meaningful it must be verifiable.)

Ex 3.8

1. (a) LNC (b) LEM (c) Com & (d) LBV (e) DeM (f) Contrap (g) DN (h) DeM (i) Assoc V (j) Exim
2. (a) (i) No. Here "and" is used in the temporal and causal sense of "and then".
   (ii) No. Here "and" is used conditionally: the sentence may be rephrased as "If you meditate then you will find peace". Perhaps there is also a command or at least strong advice being issued here.
   (b) Yes. Of course this does not change the fact that & is associative.
3 $(\sim p \,\&\, \sim q) \supset \sim(p \vee q)$; $(p \vee q) \supset \sim(\sim p \,\&\, \sim q)$; $\sim(\sim p \,\&\, \sim q) \supset (p \vee q)$
4. (a) No (inverse) (b) No (converse) (c) Yes (contrapositive) 5. Translate as $K \supset G$. Equivalent : d f, k, m, n.

Ex 4.2

1. (a) $D \supset M, H \supset M, D \vee H \;/ \therefore M$ (b) $R \supset M, M \;/ \therefore R$ (c) $N \supset F, N \;/ \therefore F$ (d) $\sim(P \,\&\, C), P \;/ \therefore \sim C$ (e) $T \supset S, \sim S \;/ \therefore \sim T$ (f) $(G \,\&\, H) \vee D, \sim(G \,\&\, H) \;/ \therefore D$ (g) $M \supset S, \sim M \;/ \therefore \sim S$ (h) $T \supset N, N \supset C, C \supset E \;/ \therefore T \supset E$ (i) $K \supset F \;/ \therefore \sim F \supset \sim K$ (j) $G \supset R, R \;/ \therefore G$ (k) $C \supset E, \sim E \;/ \therefore \sim C$ (l) $D \supset A, \sim D \;/ \therefore \sim A$ (m) $R \supset N, N \supset L, L \supset C \;/ \therefore R \supset C$ (n) $\sim(I \,\&\, A), I \;/ \therefore \sim A$ (o) $S \supset U \;/ \therefore \sim U \supset \sim S$ (p) $R \supset W, H \supset W, R \vee H \;/ \therefore W$ (q) $(R \,\&\, F) \vee D, \sim(R \,\&\, F) \;/ \therefore D$ (r) $V \supset L, V \;/ \therefore L$
2. (f) A = George will apologize and Harold will accept his apology; D = They will have a prolonged dispute. $A \vee D, \sim A \;/ \therefore D$ (q) A = The P.M. will resign and the Cabinet will fail to elect a new P.M.; D = The Senate will bring the Government down. $A \vee D, \sim A \;/ \therefore D$

Ex 4.3

1. PC-valid: a, c, d, e, f, h, i, k, m, n, o, p, q, r PC-indeterminate: b, g, j, l 2. D

Ex 4.4

1. Code: **V** = Valid; **I** = Invalid. (a) **I**: $A=0, B=1$ (b) **V** (c) **V** (d) **I**: $A=0, B=1$ (e) **I**: $A=0, B=1$ (f) **I**: $A=0, B=1$ (g) **V** (h) **I**: $A=C=1, B=0$ (i) **V** (j) **I**: $A=C=1, B=0; A=0, B=C=1; A=B=0, C=1$ (k) **I**: $A=1, B=C=D=0$
2. (a) $T \vee L, T \;/ \therefore \sim L$ **I**: $T=L=1$ (b) $T \vee L, \sim L \;/ \therefore T$ **V** (c) $\sim(T \,\&\, L) \;/ \therefore \sim T \,\&\, \sim L$ **I**: $T=1, L=0; T=0, L=1$ (d) $\sim T \vee \sim L \;/ \therefore \sim(T \vee L)$ **I**: $T=1, L=0; T=0, L=1$ (e) $S \equiv T, \sim T \vee R \;/ \therefore S \supset R$ **V** (f) $H \supset M, \sim M \vee (R \,\&\, H), R \;/ \therefore M \vee H$ **I**: $H=M=0, R=1$ (g) $A \supset N, \sim N \supset L \;/ \therefore A \vee L$ **I**: $A=L=0, N=1$ (h) $S \supset E \;/ \therefore S \vee \sim E$ **V** *($S=0, E=1$ is impossible).*
3. (a) $T \supset B, \sim T \vee W, W \;/ \therefore B$ **I**: $T=B=0, W=1$ (b) $S \vee O, S \supset \sim T, \sim T \;/ \therefore S$ **I**: $S=T=0, O=1$
   (c) $S \supset (M \supset B), \sim B \,\&\, S \;/ \therefore \sim M$ **V** (d) $(A \,\&\, R) \supset (E \supset Y) \;/ \therefore (A \,\&\, \sim Y) \supset (\sim R \vee \sim E)$ **V** (e) $W \equiv H, H \vee \sim(W \vee S) \;/ \therefore H \supset S$ **I**: $W=H=1, S=0$ (f) $A \supset I, (P \vee G) \,\&\, (G \supset I), \sim P \;/ \therefore A$ **I**: $A=P=0, I=G=1$ (g) $G \supset S, (S \supset N) \vee C, \sim C \,\&\, \sim N / \therefore \sim G$ **V** (h) $P \supset S, \sim S \supset (P \supset D), S \supset \sim D \;/ \therefore S \supset (P \,\&\, \sim D)$ **V** ($P=D=0, S=1$ is impossible).
4. (a) Yes (b) No 5. (a) $R \supset F, S \supset F, F \;/ \therefore R \vee S$ (b) $R=0, F=S=1$
6. $R=F=1, S=0; R=S=0, F=1$ Note: $R=0, F=S=1$ is impossible

Ex 4.5

1. b, c, d 2. Code: **V** = Valid; **I** = Invalid; PI = Premises Inconsistent; CN = Conclusion necessary. (a) $M \vee \sim M \;/ \therefore T$ **I** (b) $M \,\&\, \sim M \;/ \therefore T$ **V**, PI (c) $T \;/ \therefore M \vee \sim M$ **V**, CN (d) $T \;/ \therefore M \,\&\, \sim M$ **I** (e) $P, P \supset R, R \supset W, \sim W \;/ \therefore R \supset \sim W$ **V**, PI (f) $L \supset H, H \supset \sim L \;/ \therefore \sim L$ **V** (g) $L \equiv (R \,\&\, A), A \,\&\, \sim R, L \;/ \therefore L \,\&\, \sim R$ **V**, PI (h) $T \;/ \therefore S$ **V**, PI, CN (i) $J \,\&\, \sim C, I \supset C, \sim I \supset N \;/ \therefore I \vee C$ **V**, PI
3. Sound: c 4. True: a, c, d, g, i

Ex 4.6

1. Code: **V** = Valid; **I** = Invalid. (a) **V** (b) **I**: $p=0, q=1$ (c) **V** (d) **V** (e) **I**: all rows (f) **V** (g) **I**: $p=q=1, r=0; p=r=1, q=0$ (h) **V** 2. (e) 3. (a) i, iii (b) Yes
4. (a) i, ii, iv (b) i, ii are invalid, iv is valid. (c) Yes
5. $R \;/ \therefore P$ (a) No (b) Yes 6. True: a, d, e, f, g, j
7. (a) DC (or MT) (b) DN (c) DD (d) RAA (e) Contrap (f) AA (or MP) (g) Com (h) DeM (i) AA (or MP)

Puzzle 4. Cyril is *honest* because the honest person must say he is honest. Alan can't be the liar because if he was he would be telling the truth. So Alan is *ordinary*. Hence Betty is the *liar*.

Ex 5.3

1. Tautologies; a, c, d, f, g, j 2. Contrad.: a, c, e 3. (a) Taut. (b) Conting. (c) Contrad.
4. (a) $(A \not\equiv B) \,\&\, (B \,\&\, A)$ PC-contrad. (b) $\sim A \supset \sim(A \,\&\, B)$ Taut. (c) $(A \,\&\, B) \vee \sim A$ PC-indet.
5. It is a contingency 6. Code: T = Necessary Truth; F = Contradiction (a) $(I \supset S) \vee \sim Y$ T (b) $(V \,\&\, Y) \,\&\, \sim I$ F (c) $W \supset \sim(H \,\&\, M)$ T (d) $[W \,\&\, (M \supset \sim H)] \supset \sim M$ T (e) $S \,\&\, (H \supset \sim M)$ F

Ex 5.4

1. (a) Yes (b) No (c) Yes 2. (a) Yes (b) Yes (c) Yes (d) No e.g., $p=q=r=0$
(e) Yes (f) No e.g., $p=0,\ q=r=1$ 3, a, b

Ex. 5.5

1. (a) **V** (b) **I:** $p=0,\ q=1$ (c) **V** (d) **I:** $p=0,\ q=r=1$ (e) **I:** $p=s=t=0,\ r=1,\ q$ optional.
2. (a) PC-valid (b) PC-indet. (c) PC-valid (d) PC-indet.
3. (a) **V** (b) **I:** $A=0, B=1$ (c) **V** (d) **I:** $A=B=C=E=0, D=1$
4. (b) No. Premises are inconsistent.
5. (a) $C \supset (L \vee D),\ C\ /\ \therefore L \,\&\, D$ **I:** $C=L=1, D=0$ or $C=D=1, L=0$ (b) $M \supset C,\ C \supset (P \,\&\, R)$
$/\ \therefore (\sim P \vee \sim R) \supset \sim M$ **V**

Puzzle 5. If *A* is a student she will lie and say "no". If *A* is a lecturer she will tell the truth and say "No". So *A* replies "No", and consequently *B* is telling the truth and is therefore a lecturer. From what *C* says we know that if *A* is a student *C* is truthful and hence a lecturer; if *A* is a lecturer then *C* is lying and hence a student. So there are two possible arrangements, and in each of these there is exactly one student.

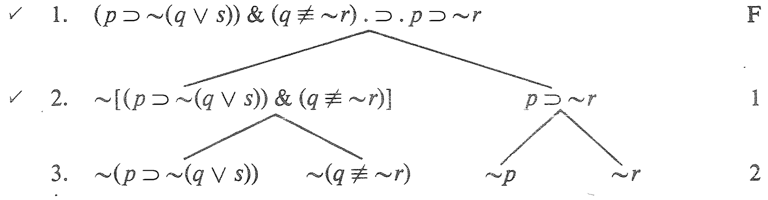
Ex 6.2

1. (a) No (b) Yes (c) No (d) No (e) No (f) Yes

Ex 6.3A

1. Contradictions: a, b, c, d, e, i, j, l. Here are some *sample solutions:*

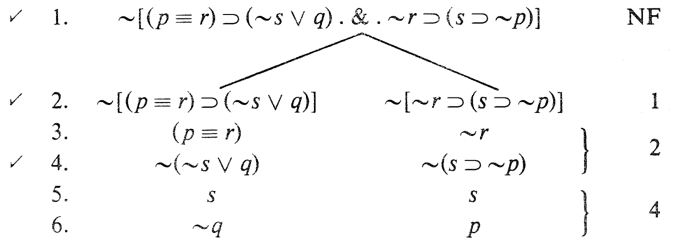
(a) ✓ 1. $\sim(p \supset p)$ F — The tree closes.
2. p } 1 — Therefore $\sim(p \supset p)$ is a *contradiction.*
3. $\sim p$
×

(k)


The two paths on the right will remain open, so we do not have to go any further. The formula is *not* a contradiction.

2. Tautologies: a, d, f, j, l, m. Here are some *sample solutions:*

(a) ✓ 1. $\sim(p \supset p)$ NF — Making the formula false leads to a contradiction. Hence $p \supset p$ must always be true: it is a *tautology.*
2. p } 1
3. $\sim p$
×

(k)


The right path will not close. Hence the formula is *not* a tautology.

Ex. 6.3B

1. (a) $p=r=0, q=1$ (b) $p=q=r=1;\ p=q=1, r=0$ (c) $p=r=0, q=1;\ p=q=r=0$ (d) None
2. Code: T = Tautology; F = Contradiction; C = Contingency.
   (a) C: $p=1; p=0$ (b) T (c) C: $p=q=0; p=1, q=0$ (d) F (e) C: $p=q=1; p=1, q=0$
   (f) C: $p=q=r=1; p=q=1, r=0$ (g) $p=q=r=s=1; p=q=r=1, s=0$
3. (a) $P \& \sim(\sim S \vee P)$ PC-contradiction (b) $R \vee (F \& C)$ PC-indeterminacy
   (c) $B \supset [C \supset (C \& B)]$ Tautology. 4. Contingency
5. (a) $J \supset \sim M$ Necessary Truth (b) $\sim J \supset M$ Contingency (c) $\sim D \& P \& E$ Contradiction
   (d) $(L \& \sim A) \supset (A \& \sim L)$ Necessary Truth

Ex. 6.4

1. (a) Yes (b) No (c) Yes 2. (a) No (b) Yes (c) Yes

Ex 6.5

Code: **V** = Valid; **I** = Invalid

2. (a) **V** (b) **I**: $p=0, q=1$ (c) **V** (d) **I**: $p=1, q=0$ (e) **I**: $p=1, q=0$ or $p=0, q=1$
   (f) **V** (g) **V** (h) **V** (i) **I**: $p=r=0, q=1$ (j) **V** (k) **V** (l) **I**: $p=q=r=s=0$ (m) **V**
6. (a) $L \supset I,\ \sim L\ /\ \therefore \sim I$ **I**: $L=0, I=1$ (b) $\sim(R \& B),\ \sim R\ /\ \therefore B$ **I**: $B=R=0$ (c) $J \supset R,\ S \supset (J \vee D), S\ /\ \therefore D \vee R$ **V** (d) $\sim(P \supset \sim Q), (P \& Q) \supset S, S \supset \sim P\ /\ \therefore \sim Q$ **V**
   (e) $S \not\equiv M,\ \sim S \supset D,\ \sim D \vee I, M \equiv I\ /\ \therefore D \vee \sim S \vee \sim I$ **V** (f) $L \vee O, M \supset S, O \vee E\ /\ \therefore (\sim S \& \sim E) \supset (L \vee M \vee \sim E)$ **V** (g) $\sim F \supset \sim P,\ \sim P \supset \sim W,\ \sim W \supset (Y \vee L), L \supset \sim I\ /\ \therefore I \supset (Y \vee F)$ **V** (h) E = I believe that God exists; T = I am a theist; A = I am an atheist; D = I believe that God doesn't exist; N = I am an agnostic (note that D does *not* mean the same as $\sim E$). $(E \supset T) \& (D \supset A), N \supset (\sim T \& \sim A)\ /\ \therefore N \supset (\sim E \& \sim D)$ **V** (i) $A \vee B \vee C \vee D \vee E, A \not\equiv B, C \equiv (D \vee E), A \vee C \vee E, (D \& B) \vee (\sim D \& \sim B)\ /\ \therefore A \& C \& E \& \sim B \& \sim D$ **I**: $A=1, B=C=D=E=0$ etc. Here is one case where a tree is less efficient than a table even though there are several propositional letters. (j) $S \supset (\sim L \vee Z),\ \sim L \supset \sim M, M \vee F,\ \sim F\ /\ \therefore Z \vee \sim S$ **V** (k) $S \supset E\ /\ \therefore S \vee \sim E$ **V** ($S=0, E=1$ is impossible).

Ex. 6.6

3. (a) $H \supset (R \vee I), H \vee D, R \equiv A,\ \sim A, H\ /\ \therefore I \vee D$ **V** (b) $B \supset E, W \vee (E \& F), L \& \sim W\ /\ \therefore F \supset (B \vee \sim L)$ **I**: $B=W=0, E=F=L=1$ (c) $C \supset [I \& (O \supset S)], W \vee \sim I, (O \supset S) \supset W\ /\ \therefore \sim(C \& \sim W)$ **V** (Note that the third premise was redundant). (d) $A \supset B,\ \sim A, K \supset (E \supset B), R \supset K,\ \sim B\ /\ \therefore \sim E \vee \sim R$ **V**

Puzzle 6 (a) BASIS BASKS TASKS (b) PATH PATS PAWS LAWS (c) CITES MITES MOTES MODES MODEL MODAL (d) THINK THINE SHINE SHONE PHONE PRONE PROVE (e) TREES FREES FRIES FLIES FLITS FLATS PLATS PLOTS BLOTS BOOTS BOOTH BROTH TROTH TRUTH

Ex 7.2

1. (a) All women are inferior. (b) Only financial members of the union will get the pay rise. (c) This argument is an enthymeme. (d) Becoming a competent logician is really worthwhile.
2. (a) Fred is unfit for your job. (b) The government member is not interested in maintaining a high standard of education in our country.
3. (a) If you were a greedy little brat you wouldn't be satisfied with your usual allowance. But I know you're not a greedy little brat. So you will be satisfied with your usual allowance, won't you? (b) Only if you are rich enough to afford it will you donate \$50 000 to charity. I can see that a fine gentleman like you is rich enough to afford it. So no doubt you will donate \$50 000 to charity.
4. Although the explicit PL-form of the argument is invalid, subatomic analysis reveals the argument is valid (the conclusion follows from the second premise).
5. (a) U = The universe exists; S = The universe had a start; C = Something caused the universe; G = God exists. $(U \supset S) \& U, C \equiv G\ /\ \therefore G$ Countermodel: $U=S=1, C=G=0$. (b) Add $S \supset C$ as a premise. This eliminates the countermodel. (c) An argument is sound iff it is valid (no logical error) and has true premises (no factual error). Thus if Mr Agnostic disagrees with any of the premises (e.g., he might believe the universe always was) he is not bound (by this argument) to accept the conclusion.

6. (a) One counterexample is a possible world where he studies Chinese but not Indian philosophy and he knows of Lao Tzu but not Shankara. A valid argument is obtained if we modify the conclusion to : Either he doesn't study Chinese philosophy or he doesn't study Indian philosophy. (b) One counterexample is a possible world where he follows Yogananda and values both Hinduism and Christianity. A valid argument is obtained if we modify the conclusion to: He values Hinduism if and only if he values Christianity.
7. A counterexample is a possible world in which almost every logic student gets married but the next logic student does not get married. An inductively strong argument is obtained if the claim of support is weakened from certainty to mere probability as follows: Almost every logic student gets married; so probably the next student to enroll in logic will get married. We now have an inductive rather than a deductive argument.

Ex 7.3

1. (a) ↔ *T & P*, (b) ↔ *T & ∼P;* (c) ← *R;* (d) → *G ⊃ ∼F;* (e) →*R ⊃ G;* (f) → *∼R ∨ G*
2. (a) ↔ *∼∼F;* (b) → *∼G;* (c) →*R & ∼G;* (d) ↔ *∼R & ∼P;* (e) →*R ∨ P or* (e) ↔*R ≢ P;* (f) →*F ∨ G;* (g) →*F ⊃ ∼G;* (h) →*∼P;* (i) ←*∼(R ⊃ F);* (j) →*∼(F & R)*

Ex 7.4

1. (a) →*R ⊃ (if F C)* (a) →*R ⊃ (F ⊃ C);* (b) →*not R ⊃ not F* (b) →*∼R ⊃ ∼F;* (c) →*R ⊃ (if not F not C)* (c) →*R ⊃ (∼F ⊃ ∼C);* (d) →*not (if R then F) ⊃ not C* (d) →*∼(if R then F) ⊃ ∼C* (d) →*∼(R ⊃ F) ⊃ ∼C;* (e) →*if R then F ⊃ C (A = if R then F)* (e) →*A ⊃ C;* (f) →*C ⊃ if R then F* (f) →*C ⊃ (R ⊃ F);* (g) →*not C ⊃ not (if R then F)* (g) →*∼C ⊃ ∼A;* (h) →*neither R nor F ⊃ not C* (h) →*∼(R ∨ F) ⊃ ∼C;* (i) ↔*∼(if F only if R then not C)* or (i) ←*∼(F only if R ⊃ not C) (B = F only if R)* (i) ←*∼(B ⊃ ∼C);* (j) ←*∼(B ⊃ C only if F)* (j) ←*∼(B ⊃ (C ⊃ F));* (k) →*if RF ∨ not C* (k) →*(R ⊃ F) ∨ ∼C;* (l) →*not if FC ∨ not if RF* (l) →*∼if FC ∨ ∼if RF (D = if FC)* (l) →*∼D ∨ ∼A*

Ex 7.5

(**A** = adequate; **I** = inadequate; **N** = Necessary Truth; **C** = Contradiction).

1. (a) ↔*L & ∼L* **A**; (b) ←*L ≢ ∼L* **A** for **N** or not **C**; (c) ↔*L ∨ ∼P* **A**; (d) →*M ⊃ ∼M* **A** for not **N** or **C**; (e) →*(L & P) ⊃ (P & L)* **A** for not **N** or **C**; (f) →*L ⊃ (P ∨ L)* **A** for not **N** or **C**.
2. (*y* = yes, it holds; *n* = no, it is not shown to hold)
   (a) *P →C ⊃ B, Q ↔∼C ∨ B*: (i) **A** *n* (ii) **A** *y* (iii) **I**
   (b) *P ↔∼(B & E), Q ↔∼B ∨ ∼E*: (i) **A** *n* (ii) **A** *y* (iii) **A** *y*
   (c) *P ↔C & B, Q ↔C & ∼B*: (i) **A** *y* (ii) **A** *n* (iii) **A** *n*
   (d) *P →C ⊃ B, Q →C ⊃ ∼B*: (i) **A** *n* (ii) **I** (iii) **I**
   (e) *P ↔B ≢ E, Q →∼B ⊃ ∼E*: (i) **A** *n* (ii) **I** (iii) **A** *n*

Ex 7.6

(1) *E ⊃ G, H ⊃ E / ∴ ∼G ⊃ ∼H* **V I**; (2) *(I ∨ W) ⊃ (M & S), ∼S / ∴ ∼W* **V A**; (3) *N ≡ Q, ∼K ∨ E / ∴ (K & (E ⊃ Q)) ⊃ N* **V I**; (4) *∼T ∨ M, (T & ∼W) ∨ ∼(M ∨ G), ∼M ⊃ ∼G, T ⊃ (T & ∼W) / ∴ W ≡ ∼T* **InV** *(W = T = M = G = 0)* **I**; (5) *A ∨ F, G ⊃ A, U ⊃ F / ∴ G ∨ U* **InV** *(G = U = 0, A = F = 1)* **I**; (6) *(G & ∼R) ⊃ H, ∼H & G / ∴ R* **VA**; (7) *∼O ∨ F, F ⊃ (C ∨ R), R ⊃ V, ∼C / ∴ O ⊃ V* **VI**; (8) *(I ⊃ B) ⊃ (A ∨ E), ∼A & I / ∴ E* **InV** *(E = A = B = 0, I = 1)* **I**; (9) *B ⊃ F / ∴ ∼F ⊃ ∼B* **VI**; (10) *M ⊃ D, ∼D / ∴ ∼M* **VA**; (11) *S / ∴ R ≢ ∼R* **VA**; (12) *∼M ⊃ S, S ⊃ P, M ⊃ T / ∴ P ∨ T* **VI**; (13) *M ⊃ ∼S, ∼S / ∴ M* **InV** *(M=S=0)* **I**; (14) *O ⊃ (E & F) / ∴ F ∨ ∼O* **VI**; (15) *I ⊃ C, I ⊃ (R ∨ S), ∼S, R / ∴ C* **InV** *(C=S=I=0, R=1)* **I**; (16) *F ⊃ ∼W, L & M, (L & M) ⊃ S, W / ∴ S & ∼F* **VA**; (17) *U ≡ E, S ⊃ (E ⊃ M), ∼M & S / ∴ ∼U* **VA**; (18) *S ⊃ D, (D & ∼W) ⊃ O, ∼W / ∴ ∼O ⊃ ∼S* **VI**; (19) *(P & R) ⊃ (A ⊃ ∼C), C, R / ∴ ∼P* **InV** *(P=C=R=1, A=0)* **I**; (20) *(R ∨ H) ⊃ ((P ∨ M) ⊃ (V & S)), ∼ (T & S) ⊃ I / ∴ I ∨ ∼(R & P)* **InV** *(I=0, R=P=H=M=V=S=T=1)* **I**; (21) *W ⊃ A, ∼R ∨ W, ∼A / ∴ ∼R* **VA**; (22) *E ⊃ ∼Q, ∼Q ⊃ A, L ⊃ Q, Q ⊃ B, E ⊃ ∼ L / ∴ A ∨ B* **VI**; (23) *(V & S) ⊃ P, ∼S / ∴ ∼P* **InV** *(P=1, V=S=0)* **I**; (24) *P ⊃ ∼C, F ⊃ C, F / ∴ ∼P* **VA**; (25) *P, B ⊃ O, (S & P) ⊃ ∼O / ∴ ∼ B ∨ ∼S* **VI**; (26) *P ∨ ∼E, P ⊃ A / ∴ ∼A ⊃ ∼E* **VI**; (27) *L ⊃ H, H ⊃ G, ∼H ⊃ ∼G / ∴ G ∨ ∼L* **VI**; (28) *C ⊃ D, ∼D / ∴ ∼C* **VA**; (29) *∼(S ∨ U) ⊃ C, S / ∴ ∼ C* **InV** *(C=S=U=1)* **I**; (30) *D ∨ E, E ⊃ I, ∼I / ∴ D* **VA**; (31) *T ⊃ I, P ⊃ I / ∴ P ⊃ T* **InV** *(P=I=1, T=0)* **I**; (32) *D ⊃ V ,∼D ⊃ H / ∴ V ∨ H* **VI**; (33) *C ⊃ R, (C & R) ⊃ W, (C ⊃ W) ⊃ ∼S, S ∨ M/ ∴ M* **VI**; (34) *(L & G) ⊃ F, F ⊃ S, G & ∼S / ∴ ∼L* **VA**; (35) *(S ∨ I) ⊃ B, R ⊃ B/ ∴ R ⊃ (S ∨ I)* **InV** *(R=B=1, S=I=0)* **I**; (36) *∼P ⊃ E, ∼E / ∴ P ∨ ∼K* **VI**; (37) *J ∨ M, ∼J, M ⊃ T, T ⊃ B / ∴ B* **VA**; (38) *P ∨ S, P ⊃ O, S ⊃ (W ∨ O), ∼W / ∴ O* **VA**; (39) *∼C ⊃ P, P ⊃ L, L ⊃ R / ∴*

$\sim R \supset C$ **VI**; (40) $O \supset W, O \supset A, W \;/\; \therefore A$ **InV** $(A=O=O, W=1)$ **I**; (41) $C \supset (A \supset R), C \,\&\, \sim R \;/\; \therefore \sim A$ **VA**; (42) $S \vee \sim F, \sim S \;/\; \therefore \sim F$ **VA**; (43) $R \supset C, C \supset S, R \;/\; \therefore S$ **VA**; (44) $G \vee L, G \supset M, \sim L \;/\; \therefore M$ **VA**; (45) $I \vee D, I \supset T, D \supset S, S \supset (M \,\&\, Q), (T \vee M) \supset P \;/\; \therefore P$ **VA**; (46) $B \supset (A \,\&\, D), A \,\&\, D \;/\; \therefore B$ **InV** $(B=O, A=D=1)$ **I**; (47) $A \vee G, G \supset \sim U, \sim U \supset \sim S \;/\; \therefore S \supset A$ **VI**; (48) $S \supset A, A \supset \sim C, C \supset \sim W \;/\; \therefore S \supset \sim\sim W$ **InV** $(S=A=1, W=C=O)$ **I**; (49) $W \supset (F \vee S) \;/\; \therefore (W \supset F) \vee (S \supset W)$ **VI**; (50) $(G \equiv T) \,\&\, \sim G, T \vee \sim R \;/\; \therefore R \supset D$ **VI**; (51) $(S \supset L) \,\&\, (H \supset \sim L), H \supset S, H \supset P \;/\; \therefore S \supset P$ **InV** $(S=L=1, P=H=O)$ **I**; (52) $R \supset J, \sim R \;/\; \therefore \sim J$ **InV** $(R=O, J=1)$ **I**; (53) $\sim(S \vee A), \sim S \supset M, \sim M \vee C \;/\; \therefore \sim A \,\&\, C$ **VA**; (54) $L \vee S, S \supset \sim I, L \supset \sim I \;/\; \therefore \sim I$ **VA**; (55) $F \supset \sim W, \sim W \;/\; \therefore F$ **InV** $(F=W=O)$ **I**; (56) $D \supset S, S \supset T, S \,\&\, \sim D \;/\; \therefore \sim T \vee \sim(D \supset S)$ **InV** $(T=S=1, D=O)$ **I**; (57) $M \vee T, M \;/\; \therefore \sim T$ **InV** $(M=T=1)$ **I**; (58) $D \supset C \;\; U \,\&\, C, \sim D \equiv U \;/\; \therefore (D \vee \sim D) \,\&\, C$ **VA**; (59) $F \supset \sim S, \sim S \;/\; \therefore F$ **InV** $(F=S=O)$ **I**; (60) $T \supset C, E \supset C, \sim E \;/\; \therefore \sim T$ **InV** $(T=E=C=O)$ **I**; (61) $\sim (D \,\&\, T), \sim T \;/\; \therefore D$ **InV** $(D=T=O)$ **A**; (62) $L \supset F, C \supset J \;/\; \therefore (L \,\&\, C) \supset J$ **VI**; (63) $C \,\&\, H, H \supset \sim D \;/\; \therefore \sim (C \,\&\, D)$ **VA**; (64) $\sim A \vee R, \sim R \,\&\, F \;/\; \therefore \sim A$ **VA**; (65) $(C \,\&\, G) \supset W \;/\; \therefore \sim C \supset \sim W$ **InV** $(C=G=O, W=1)$ **I**

Puzzle 7 The master has not contradicted himself. When the master and student utter the sentence "The oak tree is in the garden" they are asserting different propositions. The master has arrived at his answer by intuitional experience but the student has simply repeated the answer without understanding why it is an answer.

Ex 8.2

1. (a) DN (b) DN (c) Contrap (d) DeM (e) Idem (f) Dist (g) Dist (h) MI
2. Lines: 4 (3 MI); 5 (4 DeM); 6 (5 MI); 9 (8 Exim); 10 (9 Dist (and Com)); 16 (15 MI)
3. We list here the relevant Rules in the order they are used for one solution (other solutions are possible): if you are stuck, use one or more of these Rules as hints. (a) Idem, DN (b) DeM, Com, MI, (c) ED, ME, DeM, MI (d) MI, Assoc, Com, MI (e) MI, MI, DeM, DN, Com, (f) MI, Dist (and Com), DN, DeM , MI, MI.

Ex 8.3

1. (a) 1 Simp; 3 Simp; 4 Simp; 2,5 AA (b) 1 Simp; 1 Simp; 2,4 Conj; 5 Com; 4,3 AA; 7 DN; 3,8 Conj (c) 2 DN; 4 ME; 5 Simp; 1, 6 AA; 5 Simp; 3,8 AA; 7,9 AA; 10 DN (d) 1 ME; 4 Simp; 2 ME; 6 Simp (or Idem); 5,3 AA; 6 Idem (or Simp); 9,3 AA (e) 1 ME; 5 Simp; 6,2 AA; 4 ME; 8 Simp; 9,3 AA; 7,3,10 SCD (or CCD, Idem); 11 DN.
2. Relevant Rules: (a) Ch Ar, DC (b) Ch Ar, AA (c) DC, DD (d) DC, DC, Conj (e) DC, Add (f) Simp, Add (g) Contrap, Contrap, CCD (h) DN, ME, Simp, AA, Conj (i) AA, Conj, AA (j) Add, Com, AA, Add

Ex 8.4

1. (a) A; 1 Simp, 2, 3 AA; 1 Simp; 4, 5 Conj; 2–6 RAA (b) A; 1, 3 AA; 4, 2 Conj; 3–5 RAA (c) A; 3–3 CP, 4 MI; 5, 2, 1 SCD (or CCD, Idem) (d) A; 2 Addx2; 3, 4 Conj; 2–5 CP; A; 7 Simp; 8 Add, Com; 7 Simp; 10 Add, Com; 9, 11 Conj; 7–12 CP; 1, 6, 13 SCD. (e) A; 1 DeM; 3 MI; 4, 2 AA; 2 Com; 6, 5 Conj; 7 DN; 2–8 CP
2. Relevant Rules: (a) A, AA, DeM, DN, DD, CP (b) A, AA, DC, CP (c) A, AA, AA, Conj, DeM, DC, CP (d) A, Add, AA, CP (e) A, SCD, CP
3. Relevant Rules: (a) A, DC, DN, AA, Conj, RAA, DN (b) A, AA, AA, Conj, RAA (c) A, DeM, RAA, DN (d) A, DC, DN, DeM, DN, DD, Conj, RAA, DN (e) A, DeM, DN, Simp, AA, DD, Simp, Simp, Conj, RAA, DN

Ex 8.5

1. Relevant Rules: (a) A, Simp, CP (b) A, Simp, Simp, Ch Ar, CP (c) A, Simp, Simp, A, SCD, CP, CP (d) See proof of Permutation in § 8.2 example and use CP, CP, Conj, ME (e) A, ME, Simp, MI, A, DN, DD, Simp, CP, CP
2. Relevant Rules: (a) DD, ME, Simp, AA, MI, DN, DD, Conj (b) DN, DD, DeM, DN, DD, Conj (c) Simp, Simp, MI, DeM, DN, Simp, AA, Conj (d) DeM, DN, Simp, Simp, MI, DeM, DN, Simp, AA, Simp, Conj (e) ME, Simp, Simp, MI, Idem, DC, AA, Conj

Ex 8.6

1. Relevant Rules: (a) AA, AA (b) AA, Add, Com (c) ChAr, ChAr, (d) AA, DN, DC, DN, Conj (e) Simp, DC, DN, AA, Simp, AA, Idem (f) Simp, DN, DC, DC (g) DeM, Simp, DC, AA (h) Contrap, DC, AA, AA, Add (i) AA, DC, DC (j) Contrap, DeM, AA, AA, Add, Com (k) Contrap, DN, DeM, AA, AA (l) A, AA, Simp, Add, MI, CP (m) A, AA, ChAr, A, AA, Simp, CP, CP (n) A, DC, DD, Simp, Conj, RAA, DN (o) A, AA, A, AA, AA, AA, CP, CP (p) Simp, Simp, A, A, AA, Simp, Conj, AA, Simp, Conj, RAA, DN, CP (q) MI, Idem (r) MI, Idem, DC, DeM, DN, DD, DC, DeM, Com
2. Relevant Rules: (a) A, CP, MI, Com (b) A, A, theorem (a), SCD, CP, CP (c) A ($\sim p$), Add, CP, MI, MI, DN (d) A, A, DN, MI, ChAr, MI, DN, Com, CP, CP (e) Assume $\sim[(p \equiv q) \vee (q \equiv r)]$, then use DeM, Simp and ME (2nd version), DeM, Simp, DeM, DN, MI a number of times, then assume p to get $p \supset r$ and assume r to get $r \supset p$, and hence by Conj and ME get $p \equiv r$; then discharge the first assumption and use MI (and Assoc) to finish the proof.
3. Relevant Rules are listed after the translation.
   (a) $A \supset O, O \supset W \;/\; \therefore \sim W \supset \sim A$ ChAr, Contrap (b) $(S \supset B) \,\&\, (\sim S \supset E) \;/\; \therefore E \vee B$ Simp, Simp, A, CP, MI, CCD (c) $B \supset (N \vee A), N \supset L, L \supset \sim D, D \,\&\, B \;/\; \therefore A$ Simp, Simp, DN, DC, DC, AA, DD (d) $L \,\&\, (B \supset C), C \supset R, \sim R \;/\; \therefore \sim(L \supset C)$ Simp, Simp, DC, Conj, DN, DeM, MI (e) $O \supset W, W \supset T, H \equiv T, \sim H \;/\; \therefore \sim O$ ME, Simp, DC, DC, DC (f) $E \vee [\sim E \supset ((H \,\&\, W) \vee \sim W)], W \,\&\, \sim E \;/\; \therefore H$ Simp, Simp, DD, AA, DN, DD, Simp (g) $M \supset \sim M, M \;/\; \therefore \sim E$ AA, A, Conj, RAA (h) $W, (C \,\&\, W) \supset P, (P \,\&\, S) \supset (D \vee L), \sim L \,\&\, S \;/\; \therefore D \vee \sim C$ Simp, Simp, A, Conj, AA, Conj, AA, DD, CP, MI, Com

Puzzle 8 (1) Place on plank across one corner of the moat, then place the other plank at right angles to this, forming a "T" with its end on the island. You may now walk across and rescue the maiden.

(2) If the maiden is free to move there are several other solutions. For example, place one plank on the outer bank with its edge protruding a bit across the moat, then slide the other plank over the top of this to the island (holding things steady with your foot if necessary): the maiden may now walk across the top plank.

Ex 9.2

1. (a) She is neither 30 nor 40. (b) She is 40. (c) She is 30.
2. The thieves are Mr Avarice and Ms Belcher.
3. Boriss and Carveruppa are the murderers
4. (i) Lord Alpha (ii) Lords Alpha and Beta
5. Marilyn, Barbra and Tina wear a swimsuit and Eartha wears an evening gown.
6. Ann is Christian, Bill is Moslem, Cathy is Hindu and Don is Buddhist.
7. Alan's wife is Carmel, Bill's wife is Norma, Colin's wife is Karen and David's wife is Jean.

Ex 9.3

1. (a) +, x (b) +, x (c) +, –, x (d) +, –, x, ÷ (e) +, –, x, ÷
2. (a) 0, 0, 1, 1 (b) +, x (c) Yes (d) No e.g., $1 + (2 \times 3) \neq (1 + 2) \times (1 + 3)$
   (e) No e.g., $2 + 2' = 1 \rightarrow 2' = -1$ but then $2 \times 2' \neq 0$ (f) No (g) +, x (h) Yes (i) No
3. (a) $x \bullet (1 \bullet y) = y \bullet x$ (b) $x' \bullet (x + y) = x' \bullet y$ (c) $(0 \bullet y)' = 1 + y'$
4. (a) Not closed; Com; Not idem; id = 0, even numbers (b) Not closed; Not com; Not idem; id = 0; zero; negatives (c) Not closed; Com; Not idem; id = 1; squares (d) Not closed; Not com; Not idem; id = 1; one, reciprocals
5. Relevant theorems: (a) Id, Com, Id (b) Idem, Idem (c) Dist, Idem (d) Inv, Inv (e) Idem, Idem, Left Comp (f) Id, Id, df Comp
6. Not closed under $'$. $(a' = c)$. # is not idempotent $(a \# a \neq a)$. $\triangle$ is not commutative $(a \triangle b \neq b \triangle a)$.
7. Not closed under $\square$. ∇ is not idempotent. ∇ is not commutative.

Ex 9.4

1. (a) $\{2, 4, 6, 8, 10\}$ (b) $\{\}$ (c) $\{2, 6, 10\}$ (d) $\{2, 4\}$ (e) $\{4, 8, 10\}$ 2. Yes
3. (a) $\mathrm{T} \,\&\, (p \vee \mathrm{F}) \Leftrightarrow p$ $\quad \mathcal{E} \cap (A \cup \{\}) = A$ (b) $(p \,\&\, \mathrm{T}) \vee \sim(p \vee q) \Leftrightarrow p \vee \sim q$ $\quad (A \cap \mathcal{E}) \cup (A \cup B)' = A \cup B'$
4. (a) Non-theorem (b) Theorem (c) Non-theorem (d) Theorem

6. Relevant theorems: (a) LNC, Id, DN (b) Id, Abs (c) DeM, Assoc (d) DeM, DeM (e) DeM, Dist, LEM, Id (f) Dist, LNC, Id (g) DeM, DN, Id, Com (h) Dist, Abs
   (i) Com, Dist, LEM, Id (j) CF, Id, DeM, Abs (k) Dist, Dist, LEM, Id, Com, Dist, Com, Assoc

Ex 9.5

1. Simplified formulae: (a) A (b) F (c) $A \,\&\, B \,\&\, C$ (d) T (e) $A \vee B$ (f) A (g) T (h) $A \,\&\, (B \vee C)$ (i) $A \,\&\, B \,\&\, (C \vee D)$ (j) $A \,\&\, B \,\&\, D$ (k) $A \,\&\, B$ (l) $A \vee (B \,\&\, C)$ (m) $[(A \,\&\, B) \vee C] \,\&\, (A \vee D)$ Doesn't simplify (n) B (o) $A \vee \sim B$ (p) $\sim A \,\&\, \sim B$ (q) A (r) $B \vee (A \,\&\, C)$ (s) $(A \,\&\, B) \vee C$ (t) $A \,\&\, (C \vee D)$ (u) $\sim C \,\&\, [(A \,\&\, B) \vee (\sim A \,\&\, \sim B)]$ (v) Note that there are six pathways from left to right but only two of these are possible (any series pathways with complementary switches will not transmit). Applying Idempotence we obtain: $(A \,\&\, \sim C \,\&\, \sim B) \vee (B \,\&\, C \,\&\, \sim A)$.
2. Formula: $(A \,\&\, [(B \,\&\, (C \vee D)) \vee (C \,\&\, D)]) \vee (B \,\&\, C \,\&\, D)$
3. Formula: $A \,\&\, ([B \,\&\, (C \vee D)] \vee (C \,\&\, D))$
4. Formula: $A \,\&\, [(B \,\&\, (C \vee D \vee E)) \vee (C \,\&\, (D \vee E))$ or $[(A\&C) \,\&(B \vee D)] \vee [(B\&D) \,\&\, (A \vee C)]$
5. Formula: $\sim B$ 6. Formula: $C \,\&\, (A \vee \sim B)$
7. Formula: $D \,\&\, [(A \,\&\, [(B \,\&\, C) \vee (\sim B \,\&\, \sim C)]) \vee (\sim A \,\&\, [(B \,\&\, \sim C) \vee (\sim B \,\&\, C)])]$
9. Simplified Formulae: (a) $\sim p$ (b) $\sim p$ (c) $p \downarrow q$ (d) $p \,\&\, q$ (e) $p \,\&\, \sim q$ (f) $p \mid q$ (g) $p \,\&\, (q \vee r)$ (h) $(p \downarrow q) \,\&\, r$

Ex 9.6

1. Tautologies: a, b, c, d, h, j Contradictions: e, f Contingencies: g, i
2. Main-column values listed top to bottom are. 11100010
3. (a) Contingent (b) Contingent (c) Tautology (d) Tautology
4. (a) $p \mid (q \mid q)$; $[(p \downarrow p) \downarrow q] \downarrow [(p \downarrow p) \downarrow q]$
   (b) – (e) These are fairly tedious. Make use of ME, ED and MI.
5. (a) V, $\subset$, $\supset$, $\equiv$ (b) V, $\vee$, $\equiv$, &, $\mid$, $\not\equiv$, $\downarrow$, F (c) V, $\vee$, $\nleftarrow$, $\nrightarrow$, $\not\equiv$, &, $\not\equiv$, F

Ex 9.7

1. (a) $(\sim p \vee q) \equiv (p \supset q)$ (b) $(((p \supset q) \supset p) \supset p) \equiv (p \vee \sim p)$
   (c) $\sim(p \supset (q \,\&\, r)) \equiv ((p \,\&\, q) \supset (r \vee q))$ (d) $((p \supset q) \supset (q \supset p)) \equiv (r \vee s)$
   (e) $\sim((p \supset q) \vee (q \supset p))$
2. (a) $(\sim p \,\&\, \sim q) \supset (\sim p \vee q)$ (b) $(p \,\&\, q) \equiv (q \,\&\, p)$ (c) $((p \vee q) \not\equiv (r \,\&\, q)) \supset s$
   (d) $\sim(p \vee q) \equiv (\sim p \,\&\, \sim q)$ (e) $((p \supset q) \equiv r) \equiv s$

Ex 9.8

1. (a) *CpCqp* (b) *CKpqApq* (c) *EANpqCpq* (d) *ENApqKNpNq* (e) *CCpCpqq*
2. (a) $((p \supset q) \,\&\, (q \supset r)) \supset (p \supset r)$ (b) $((p \vee q) \supset r) \supset ((p \supset r) \,\&\, (q \supset r))$
   (c) $((p \supset q) \supset r) \supset ((r \supset p) \supset p)$ (d) $(p \supset q) \supset (\sim q \supset \sim p)$
   (e) $((p \not\equiv q) \not\equiv r) \,\&\, \sim((p \,\&\, q) \,\&\, r) \equiv (((p \vee q) \vee r) \,\&\, (\sim(p \,\&\, q) \,\&\, (\sim(q \,\&\, r) \,\&\, \sim(p \,\&\, r))))$

Puzzle 9

S	E	M	A	N	T	I	C	S	■	D	N
■	N	O	T	■	T	■	O	■	R	A	A
■	T	R	E	E	■	A	N	D	■	■	N
■	H	O	■	R	E	■	S	O	U	N	D
S	Y	N	T	A	X	■	E	T	■	O	■
■	M	■	R	■	P	■	Q	■	■	R	E
R	E	■	I	D	■	■	U	■	R	■	■
■	M	P	■	I	T	■	E	G	O	■	I
D	E	C	I	S	I	O	N	■	W	F	F
■	S	■	Q	T	■	■	T	T	S	■	F

Ex 10.2

1. Code: PN = Proper Name; SP = Singular Pronoun; DD = Definite Description
   (a) James: PN (b) Susan: PN (c) Jan, Jill: PN (d) The principal: DD (e) The Premier: DD (f) Kirsty: PN, the prize: DD (g) She, him: SP (h) You, me, him: SP, the money: DD (i) The Wizard of Id: DD (j) The economy: DD
2. (a) Falling down (b) being unsteady (c) being solicitous (d) being reassured (e) being beautiful (f) having a splitting headache (g) being deft with bandages (h) having gone to bed (i) being happy (j) being asleep
3. a, f; b, h; c, e; d, i; g, j   4. a, f; b, h; c, g; d, e; i, j

Ex 10.3

1. (a) 1. Fx B2M; 2. Ga B2M; 3. $(Fx \supset Ga)$ 1,2 R$\supset$
2. (a) (i) Fx (ii) $(Fx \vee (\exists y)\, Gy)$ (iii) $(Fx \supset Gx)$ (iv) Fx (v) $(p \,\&\, Gx)$
   (b) (i) Fy (ii) p (iii) Fy (iv) $\sim(Fy \vee Gy)$ (v) $(Fy \supset Gx)$ 3. a, d

Ex 10.4

1. (a) Alan is a farmer and Carol is a student. True. (b) Bert is not a grocer and he tries hard.True (c) If Dana is a brilliant student then Dana is a student. True (d) Although not a brilliant student, Carol is a student who tries hard. True (e) If Dana is a farmer or a grocer then Dana tries hard. False (f) Dana is a grocer if and only if Bert is a student. True (g) Alan is a farmer or Carol is a grocer, but not both. True (h) Bert and Dana are brilliant students, but Bert tries hard and Dana doesn't. True (i) If Alan is a brilliant student then he is a student, and the same goes for Bert, Carol and Dana. True (j) At least one of Alan, Bert, Carol and Dana is a farmer. True.
2.

	F	*G*	*S*	*T*	*B*
a	0	1	0	1	0
b	0	0	1	0	1
c	0	1	0	1	0
d	1	0	0	1	0

Ex 10.5A

1. (a) $(\forall x)\, Px$ (b) $(\forall x)\, Mx$ (c) $(\forall x)\,(Mx \vee Px)$ (d) $(\forall x)\,(Tx \supset Mx)$
   (e) $(\forall x)\,(Px \supset Sx)$ (f) $(\forall x) \sim Px$ (g) $(\forall x)\,(Mx \supset \sim Px)$
   (h) $(\forall x)\,(Px \,\&\, Sx)$ (i) $(\forall x)\,(Tx \supset \sim Px)$ (j) $(\forall x)\,(Sx \supset Px)$
2. (a) $Fa \,\&\, Fb$: 1,0 (b) $\sim Ga \,\&\, \sim Gb$: 1,0 (c) $(Fa \supset Ga) \,\&\, (Fb \supset Gb)$: 0,1
   (d) $(Fa \supset \sim Ga) \,\&\, (Fb \supset \sim Gb)$: 1,0 (e) $(Fa \vee Ga) \,\&\, (Fb \vee Gb)$: 1,1
   (f) $\sim(Fa \vee Ga) \,\&\, \sim(Fb \vee Gb)$: 0,0 (g) $(Fa \,\&\, \sim Ga) \,\&\, (Fb \,\&\, \sim Gb)$: 1,0
3. (a) $Sa \,\&\, Sb$ (b) $\sim Sa \,\&\, \sim Sb \,\&\, \sim Sc$ (c) $(Fa \supset Ga) \,\&\, (Fb \supset Gb) \,\&\, (Fc \supset Gc)$
   (d) Same as (c) (e) $Ta \,\&\, Tb \,\&\, Tc \,\&\, Td$
4. (a) $(Fc \vee Gc) \,\&\, (Fd \vee Gd)$: 1 $(Fb \vee Gb) \,\&\, (Fe \vee Ge) \,\&\, (Ff \vee Gf)$: 0
   (b) $(Fc \supset Gc) \,\&\, (Fd \supset Gd)$: 1 $(Fb \supset Gb) \,\&\, (Fe \supset Ge) \,\&\, (Ff \supset Gf)$: 1
   (c) $(Fc \supset \sim Gc) \,\&\, (Fd \supset \sim Gd)$: 1. $(Fb \supset \sim Gb) \,\&\, (Fe \supset \sim Ge) \,\&\, (Ff \supset \sim Gf)$:0
   (d) $\sim Fc \,\&\, \sim Fd$: 0. $\sim Fb \,\&\, \sim Fe \,\&\, \sim Ff$: 0
   (e) $(Fc \equiv Gc) \,\&\, (Fd \equiv Gd)$: 0. $(Fb \equiv Gb) \,\&\, (Fe \equiv Ge) \,\&\, (Ff \equiv Gf)$: 1

Ex 10.5B

1. (a) $(\exists x)\, Tx$ (b) $(\exists x) \sim Sx$ (c) $(\exists x)\,(Mx \vee Px)$ (d) $(\exists x)\,(Sx \,\&\, Px)$ (e) $(\exists x)\,(Sx \,\&\, \sim Px)$
2. (a) $\sim Fa \vee \sim Fb$: 0,1 (b) $(Fa \,\&\, Ga) \vee (Fb \,\&\, Gb)$: 0,1
   (c) $(Fa \vee Ga) \vee (Fb \vee Gb)$: 1,1 (d) $(Fa \supset Ga) \vee (Fb \supset Gb)$: 0,1
   (e) $(Fa \,\&\, \sim Ga) \vee (Fb \,\&\, \sim Gb)$: 1,0 (f) $\sim(Fa \vee Ga) \vee \sim(Fb \vee Gb)$: 0,0
   (g) $\sim (Fa \supset \sim Ga) \vee \sim(Fb \supset \sim Gb)$: 0, 1
3. (a) $\sim Sa \vee \sim Sb \vee \sim Sc$ (b) $(Sa \,\&\, Ta) \vee (Sb \,\&\, Tb) \vee (Sc \,\&\, Tc)$
   (c) $(Sa \,\&\, \sim Ta) \vee (Sb \,\&\, \sim Tb)$ (d) $\sim(Sa \supset \sim Ta) \vee \sim(Sb \supset \sim Tb) \vee \sim(Sc \supset \sim Tc)$
   (e) $(Sa \,\&\, Fa) \vee (Sb \,\&\, Fb) \vee (Sc \,\&\, Fc) \vee (Sd \,\&\, Fd)$
4. (a) $Fa \vee Fb \vee Fc$: 1. $Fa \vee Fe$: 0 (b) $\sim Ga \vee \sim Gb \vee \sim Gc$: 1. $\sim Ga \vee \sim Ge$:1
   (c) $(Fa \vee Ga) \vee (Fb \vee Gb) \vee (Fc \vee Gc)$: 1. $(Fa \vee Ga) \vee (Fe \vee Ge)$: 1
   (d) $(Fa \not\equiv Ga) \vee (Fb \not\equiv Gb) \vee (Fc \not\equiv Gc)$: 1. $(Fa \not\equiv Ga) \vee (Fe \not\equiv Ge)$: 1
   (e) $(Fa \,\&\, \sim Ga) \vee (Fb \,\&\, \sim Gb) \vee (Fc \,\&\, \sim Gc)$: 1. $(Fa \,\&\, \sim Ga) \vee (Fe \,\&\, \sim Ge)$: 0

Ex 10.5C

1. (a) Sa (b) $\sim Sb$ (c) $Sa \supset (\exists x)\, Sx$ (d) $\sim Sb \supset \sim(\forall x)\, Sx$ (e) $(\forall x)\, (Ix \supset Mx)$
   (f) $(\exists x) Mx \,\&\, (\exists x) \sim Mx$ (g) $Pa \supset \sim Ia$ (h) $\sim(\forall x)\, (Ix \supset Mx)$ (i) $\sim(\forall x)\, (Ix \supset \sim Px)$
   (j) $Sb \vee \sim(\forall x)\, Sx$
2. (a) $\sim(\sim Fa \vee \sim Fb)$ (b) $\sim(\sim Fa \,\&\, \sim Fb)$ (c) $\sim\sim Fa$ (d) $\sim\sim Fa$
   (e) $Gb \supset Gb$ (f) $(Gb \vee Gc) \supset (Gb \,\&\, Gc)$ (g) $(\sim Ga \,\&\, \sim Gb) \supset (\sim Ga \vee \sim Gb)$
   (h) $[(Ga \supset Fa) \,\&\, (Gb \supset Fb)] \supset [(\sim Fa \supset \sim Ga) \,\&\, (\sim Fb \supset \sim Gb)]$
   (i) $[(Ga \supset Fa) \,\&\, (Gb \supset Fb)] \supset [(Ga \,\&\, Fa) \vee (Gb \,\&\, Fb)]$
   (j) $(Fa \,\&\, Ga) \vee (Fb \,\&\, Ga) \vee (Fc \,\&\, Ga)$ (k) $(Fa \vee Fb \vee Fc) \,\&\, Ga$
   (l) $(Fa \supset Ga) \,\&\, (Fb \supset Ga)$ (m) $(Fa \,\&\, Fb) \supset Ga$
   (n) $\sim[(Fa \,\&\, Fb) \supset (Ga \vee Gb)]$ (o) $\sim([(Fb \,\&\, Gb) \vee (Fc \,\&\, Gc)] \,\&\, (\sim Fb \,\&\, \sim Fc))$
3. (a) $(Fb \vee Ga) \vee (Fc \vee Ga)$: 1 (b) $(Fb \vee Ga) \,\&\, (Fc \vee Ga)$: 1 iff $a = c$
   (c) $(Fb \vee Gc) \vee (Fc \vee Gc)$: 1 (d) $Gb \,\&\, [(Fb \vee Fa) \vee (Fc \vee Fa)]$: 0
   (e) $(\sim Fb \supset Fa) \,\&\, (\sim Fc \supset Fa)$: 1 iff $a = b$

Puzzle 10 Hint: List the 8 colour arrangements BBB through GGG and eliminate 4 of these using the information given. (The answer is Blue).

Note: For ease of typesetting we will use U for the universal quantifier $\forall$ and E for the existential quantifier $\exists$.

Ex 11.1

1. (a) $(Ex)\, Ox$ (b) $(Ex) \sim Ox$ (c) $\sim(Ux)\, (Px \supset Ox)$ (d) $(Ex)\, (Px \,\&\, Ox)$
   (e) $(Ex)\, (Px \,\&\, Bx \,\&\, Ox)$ (f) $(Ux)\, (Px \supset Bx)$ (g) $(Ux)\, [Px \supset (Bx \,\&\, Ox)]$
   (h) $(Ux)\, [(Px \,\&\, Bx) \supset Ox]$ (i) $(Ux)[(Px \,\&\, Bx) \supset \sim Ox]$
   (j) $(Ux)\, [(Px \,\&\, Ox) \supset \sim Ix]$ (k) $(Ex)\, [Px \,\&\, Ix \,\&\, (Bx \vee Ox)]$ (l) $(Ex)\, (Mx \,\&\, Ix)$
   (m) $(Ex)\, (Mx \,\&\, Ox \,\&\, \sim Ix)$ (n) $(Ex)\, (Ix \,\&\, Mx \,\&\, Px)$ (o) $(Ux)\, (Mx \supset \sim Bx)$
2. (a) $(Ex)\, (Fx \,\&\, Dx)$ (b) $(Ux)\, (Ax \supset Dx)$ (c) $(Ex)\, (Vx \,\&\, \sim Dx)$
   (d) $(Ex)\, (Vx \,\&\, \sim Dx \,\&\, Nx)$ (e) $(Ux)\, (Ax \supset \sim Vx)$ (f) $(Ux)\, [Ox \supset (Dx \,\&\, Nx)]$
   (g) $(Ex)\, (Vx \,\&\, \sim Nx)$ (h) $\sim(Ux)\, (Gx \supset Nx)$ (i) $(Ux)\, [(Ax \,\&\, Dx) \supset Gx]$
   (j) $(Ux)\, [(Fx \,\&\, Jx) \supset \sim Vx]$ (k) $(Ux)\, [(Ax \vee Ox) \supset (Dx \,\&\, Nx)]$
   (l) $(Ux)\, [(Vx \,\&\, \sim Gx) \supset Nx]$ (m) $(Ux)\, [(Fx \,\&\, \sim Ax \,\&\, \sim Ox) \supset (Jx \,\&\, Dx \,\&\, Nx)]$
   (n) $(Ux)\, (Dx \supset Gx)$ (o) $(Ex)\, (Fx \,\&\, Jx) \,\&\, (Ex)\, (Fx \,\&\, \sim Jx)$
3. (a) $(Ux)\, [Px \supset (Tx \vee Fx)]$ (b) $(Ux)\, (Tx \equiv \sim Fx)$ (c) $(Ux)\, [Px \supset \sim(Tx \,\&\, Fx)]$
   (d) $(Ex)\, (Px \,\&\, \sim Tx \,\&\, \sim Fx)$ (e) $(Ex)\, (Px \,\&\, Tx \,\&\, \sim Nx)$
   (f) $(Ux)\, [(Px \,\&\, Tx \,\&\, \sim Nx) \supset Cx]$ (g) $(Ux)\, [Wx \supset (Tx \,\&\, Px)]$
   (h) $(Ux)\, [(Px \,\&\, Wx) \supset Tx]$ (i) $(Ux)\, [(Nx \,\&\, Px) \supset \sim Cx]$
   (j) $(Ux)\, [(Px \,\&\, \sim Nx) \supset (Cx \vee Fx)]$

Ex. 11.2

1. (a) Fx (b) $(Fx \vee (Ey)\, Gy)$ (c) $(Fx \supset (Uy)\, Fy)$
2. (a) $(Hx \supset p)$ (b) Hx (c) $(Lx \,\&\, Lz)$
3. (a) $(E\underbrace{x)\, (Fx}\underbrace{\,\equiv Gx})$ (e) $(U\underbrace{y)\, [Hy} \supset (U\underbrace{y)\, Gy}]$ (h) $(E\underbrace{x)\, [(Fy \supset Gx}) \equiv (U\underbrace{x)\, Fx}]$
4. (a) $F\underline{x}$ (c) $(Ux)\, Gx \,\&\, F\underline{x}$ (g) $(Ex)\, ((Uy)\, Gy \,\&\, (Gx \,\&\, F\underline{y}))$
5. (a) $(Ux)\, Fx;\ (Ex)\, Fx$ (f) $(Ux)\, (Uy)\, (Fx \supset Gy);\ (Ex)\, (Ey)\, (Fx \supset Gy)$
6. Vacuous: (a) (Ex) (b) First (Uz) (c) $(Uz)\, (Ey)$ (d) (Ux) (e) (Ux)

Ex 11.3

1. MQT-Necessities: a, b, c, d, e
2. Counterexamples: b, d
3. Possible worlds: a, b, d, f
4. If every expansion of an MQL-form up to 2^n items is a PC-contradiction, then every finite expansion will be a PC-contradiction.

Ex 11.4

1. (a) **I**: $Fa = Ga = 1,\ Ha = 0$ (b) **I**: $Fa = 0,\ Ga = Ha = 1$ (c) **I**: $Fa = Gb = 1,\ Fb = Ga = 0$
   (d) **I**: $Fa = 1,\ Ga = Fb = Gb = 0$ (e) **V** (f) **V** (g) **V** (h) **V** (i) **I**: $Fa = Ga = Ha = 0$ (j) **V**

2. (a) **V** (b) **V** (c) **I:** $Ca = 0, Sa = Ha = 1$ (d) **V** (e) **V** Note: it is reasonable to assume that Jane and Susan are people. Also, the sentence "Susan's vehicle is registered in July" indicates that Susan has only one vehicle, and that it was registered in neither April nor May.

Ex 11.5

1. (a) $(Ux) \sim Px$ (b) $(Ex)\ Kx$ (c) $(Ux)\ (Tx \supset Kx)$ (d) $(Ex)\ Tx \ \&\ (Ex) \sim Tx$
   (e) $(Ux)\ (Px \supset Kx)$ (f) $(Ux)\ Mx$ (g) $(Ex)\ (Mx \ \&\ Sx \ \&\ Kx)$ (h) $(Ux)\ (\sim Kx \supset Mx)$
   (i) $(Ux)\ (Ax \supset Kx)$ (j) $(Ux)\ (\sim Tx \supset \sim Sx)$
2. Delete all reference to *Px* from answers to Ex 11.1 Q. 3.
3. Valid, but still need the information $\sim (As \vee Ms)$

Ex 11.6

1. (a) $P \supset [(Fa \ \&\ Ga) \vee (Fb \ \&\ Gb)]$ (b) $[P \supset (Ga \ \&\ Fa)] \vee [P \supset (Gb \ \&\ Fb)]$
   (c) $[(Ga \supset Fa) \ \&\ (Gb \supset Gb)] \supset Q$ (d) $[(Ga \supset Fa) \supset Q] \ \&\ [(Gb \supset Fb) \supset Q]$
   (e) $Ga \supset (Ga \ \&\ Gb)$ (f) $([P \supset (Ga \ \&\ Fa)] \vee [P \supset (Ga \ \&\ Fb)] \ \&\ ([P \supset (Gb \ \&\ Fa)] \vee [P \supset (Gb \ \&\ Fb)]$ (g) $[(P \supset (Ga \vee Ga)) \supset Fa] \ \&\ [(P \supset (Ga \vee Gb)) \supset Fb]$
   (h) $[(Fa \vee Fb) \supset p] \equiv [(Fa \supset p) \ \&\ (Fb \supset p)]$ (i) $[(Fa \ \&\ Fb) \supset p] \equiv [(Fa \supset p) \vee (Fb \supset p)]$
   (j) $[p \supset (Fa \ \&\ Fb)] \equiv [(p \supset Fa) \ \&\ (p \supset Fb)]$
2. (a) $T \supset (Ux)\ (Sx \supset Gx)$ (b) $\sim T \supset (Ex)\ (Sx \ \&\ \sim Gx)$ (c) $T \supset \sim Dl$
   (d) $(Ex)\ (Px \ \&\ \sim Sx \ \&\ Gx)$ (e) $(Ux)\ (Gx \supset Dx) \supset \sim T$
3 MQT-Necessities, a, c d, e (b) $Fa = 1, Fb = 0$ (f) $Ga = 1, Gb = 0$
4 Counterexamples: a, b
5. (a) $Fa = 1, Ga = 0$ (b) $Fa = Gb = 1, Ga = Fb = 0$ (c) $Fa = 0, Ga = 1$ (d) $Fa = 1$ (e) $Fa = Ga = 0$

Puzzle 11 Ask the question: "What would your twin say if I asked him whether the left door led to the men's room?" If the answer is "No", the left door does lead to the men's room; if the answer is "Yes", the right door leads to the men's room.

Ex 12.1

1. Yes: a, b, e, g, h, i, l, o No: c, d, f, j, k, m, n, p
2. Code: L = left branch; R = right branch
   (a) *Fa* UI (b) *L:* $\sim (Ex)\ Fx$ *R:* $\sim Ga$ PC (c) $Fa \ \&\ Ga$ EI (d) $Fb \vee Ga$ EI
   (e) *L:* $\sim (Ux)\ Fx$ *R: p* PC (f) $\sim (p \supset Ga)$ UI (g) *L:* $(Ex)\ Fx$ *R:* $(Ex)\ Gx$ PC
   (h) $Fa \vee (Ux)\ Gx$ EI (i) $(Ex) \sim (Fx \supset Gx)$ QN (j) $(Fb \vee Gb) \ \&\ Fa$ EI

Ex 12.2

1. (a) 1 PC; 3 QN; 2 EI; 4 UI; 6 PC (b) 1 PC; 3 QN; 4 EI; 5 PC; 2 UI
   (b) 1 PC; 3 PC; 5 QN; 2 EI; 4 EI; 6 EI (d) 1 PC; 3 PC; 5 QN; 6 EI; 2 UI; 4 UI; 8 PC
   (e) 1 PC; 3 PC; 6 QN, 7 EI; 2 UI; 9 PC (f) 2 QN; 3 UI
   (g) 3 QN; 4 EI; 5 PC; 1 UI; 8 PC; 2 UI; 10 PC (h) 2 PC; 4 QN; 1 EI; 3 UI; 5 UI; 6 PC
   (i) 3 QN; 2 EI; 5 PC; 4 UI; 8 PC; 1 UI; 10 PC; 11b PC
   (j) 3 PC; 5 QN; 6 EI; 7 PC; 4 UI; 1 UI; 2 UI; 11 PC; 12 PC; 10 PC
2. Correct: lines 1, 2, 5, 8 Incorrect: lines 3, 4, 6, 7 Tick: lines 2, 4, 5, 7
3. (j)

✓	1.	$\sim(\exists x)[(\exists x)Fx \supset Fx]$	NF
ba \	2.	$(\forall x) \sim[(\exists x)Fx \supset Fx]$	1, QN
✓	3.	$\sim[(\exists x)Fx \supset Fa]$	2, UI
✓	4.	$(\exists x)Fx$	3, PC
	5.	$\sim Fa$	
	6.	Fb	4, EI
✓	7.	$\sim[(\exists x)Fx \supset Fb]$	2, UI
	8.	$(\exists x)Fx$	7, PC
	9.	$\sim Fb$	
		X	Closure. ∴ MQT-Necessary

5. (a)

✓	1.	$(\exists x)Fx$	P
✓	2.	$\sim(\exists x)(Fx \vee Gx)$	NC
a \	3.	$(\forall x)\sim(Fx \vee Gx)$	2, QN
	4.	Fa	1, EI
✓	5.	$\sim(Fa \vee Ga)$	3, UI
	6.	$\sim Fa$	5, PC
	7.	$\sim Ga$ ×	

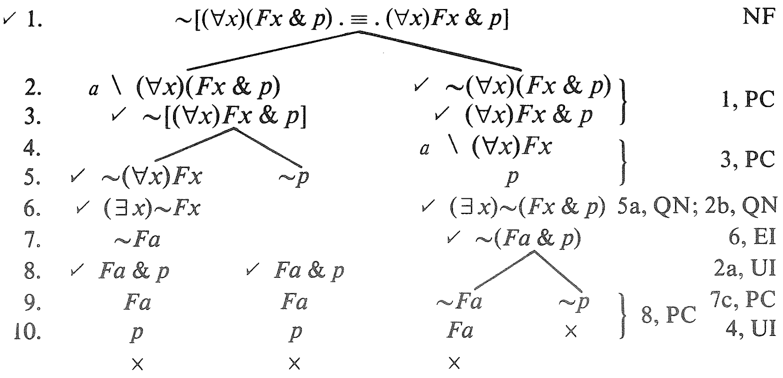
The tree closes. Hence it's impossible to have all the premises true and the conclusion false. Therefore the argument-form is valid.

6. (a) *(Ex) (Sx & Ax), (Ux) (Sx ⊃ Mx) & (Ux) [(Sx & Ax) ⊃ ∼Ex] / ∴ ∼ (Ux) (Mx ⊃ Ex)* **V** (b) *Pa & La / ∴ (Ex) (Lx & Px)* **V** (c) *(Ux) [Nx ⊃ (Tx ⊃ (Ox ∨ Ex)], (Ux) [(Nx & Hx) ⊃ ∼ (Ox ∨ Ex)] / ∴ (Ux) [Hx ⊃ (Nx ⊃ ∼Tx)]* **V** (d) Universe = persons. *(Ux) [Px ⊃ (Gx ∨ Tx)], ∼Tj & ∼ Aj & Pj / ∴ Gj* **V** (e) *(Ex) (Ix & Rx), (Ux) (Ex ⊃ Rx) / ∴ (Ex) (Ix & Ex)* **I**: *Ia = Ra = 1, Ea = 0* (f) *(Ux) (Mx ⊃ ∼Px) & (Ux) (Rx ⊃ ∼Mx) / ∴ (Ux) (Rx ⊃ Px)* **I**: *Ma=Pa=0, Ra=1* (g) Universe = persons. *(Ux) [(Sx & Lx) ⊃ Wx], (Ux) (Wx ⊃ Px), (Ex) (Sx & ∼Px) / ∴ (Ex) (Sx & ∼Lx)* **V** (h) Universe = persons. *(Ux) (Lx ⊃ Fx), (Ux) (Lx ⊃ Ex), (Ex) (Fx & Ix) / ∴ (Ex) (Fx & Ix & Ex)* **I**: *Fa=Ia=1, La=Ea=0* (i) Universe = persons. *(Ux) [Fx ⊃ (Lx & Wx)], (Ux) [(Lx & Wx) ⊃ Ex] / ∴ (Ex) (Ex & Fx)* **I**: *Fa=La=Wa=Ea=0* (j) *(Ux) (Dx ⊃ Bx), (Ux) (Bx ⊃ Wx) / ∴ (Ux) (Dx ⊃ Wx)* **V**

Ex 12.3

2. (a) True (b) False (c) True (d) True (e) True

solution for (d):



All paths close. Hence the formulae are MQT-equivalent.

Ex 12.4

2. (a) *(Ex) Fx ⊃ p* (b) *(Ex) Fx ⊃ (Uy) Gy* (c) *(Ux) Fx & (Ey) Gy* (d) *p ⊃ [(Ux) Gx ∨ (Ey) Fy]* (e) *(Ex) Fx ⊃ (Ey) Gy* (f) *(Ga & Fa) ⊃ (Ex) (Gx & Fx)* (g) *(Ux) (Fx ⊃ Gx) ⊃ (Fa ⊃ Ga)* (h) Already in MSF (i) *((Ex) Fx & Fa) ⊃ ((Ux) Fx & Fa)* (j) *(p & Ga) ⊃ [Uz) Gz ∨ ((Ux) Gx & (Uy) Fy)]*
3. Equivalent: a, c, h, i. Equivalent only when closed: d, e, j. Note: closure equivalences should not be used in substitution.
4. (a) *(Ux) (Fx ⊃ p)* (b) *(Ux) (Ey) (Fx ⊃ Gy)* (c) *(Ux) (Uy) (Gy ⊃ Fx)* (d) *(Ux) (Uy) (Gy & Fx)* (e) *(Ex) (Uy) (Gy ∨ Fx)* (f) *(Ex) (Uy) (Ew) (Uz) [(Fz ⊃ Fw) & (Fy ⊃ Fx)]*

Puzzle 12 The pilot is Jones.

13.

Answers requiring a diagram will refer to Figure *i, Fi,* in Section 13.2, and: *n=S* means *n* is shaded; $n = \alpha$ means an α in *n*; $n-m=\alpha$ means a disjunction bar from *n* to *m* with an α at each point; $n-m = \alpha - \beta$ means a bar with α and β at ends *n* and *m* respectively.

Ex 13.2

1. (a) *F1, F2: F=S;* (b) *F3, F4: 2=S;* (c) *F5, F6:* × *=2;* (d) *F3, F4: 3–1–2=*×; (e) *F3, F4: 4=S;* (f) *F5, F6: 1–2–6=*×; (g) *F5, F6: 1–2–3=*×; (h) *F5, F6: 1=S*; (i) *F5, F6: 1=5=6=S;* (j) *F5, F6: 1=2=6=S;* (k) *F3, F4: 1–3=a, 1–2=b;* (l) *F1, F2: F–F′ = a–0;* (m) *F1, F2: F–F′ =* × *– 0;* (n) *F1, F2: F = x, F′ = S;* (o) *F1, F2: F=*×*, F′ =* ×.
2. (a) *F3, F4: 1=3=4=S;* (b) *F3, F4: 3=S, 1=S;* (c) inconsistent; (d) *F5, F6: 2–6=a, 1–2–3–4=b;* (e) *F3, F4: 3–1–2=a–ab–b;* (f) *F3, F4: 2–4=a, 3–4=b;* (g) inconsistent.

Ex 13.3

1. (a) *F5, F6: 2=6=S, 3=4=S,* **V**; (b) *F3, F4:* **I**, *3=*×*, 2=*×; (c) *F3, F4: 3=S,* **V**; (d) *F5, F6: 5=6=S,* **V**; (e) *F5, F6:* **I**, *5=6=S, 1=* ×; (f) *F3, F4: 1=*×, **V**; (g) *F3, F4: 1–3=*×*, 3=S,* **V**; (h) *F5, F6: 1–2=*×*, 1=4=S,* **V**; (i) *F5, F6: 5–6=*×, **V**; (j) *F5, F6: 2–6 =* ×*, 5=6=S,* **V**; (k) *F5, F6:* **I**, *1=2 =S, 6=*×; (l) *F5, F6: 1–2–5–6=*×*, 1=4=S, 2=6=S,* **V**; (m) *F3, F4: 1–3=*×*, 1–2 =* ×, **V**; (n) *F3, F4: 3=S, 2=4=S,* **V**; (o) *F7, F8 (J=I): 1–2–5–6=*×, **V**.
2. (a) *(Ux) (Wx ⊃ Cx), (Ex)(Ux & Wx) / ∴ (Ex) (Cx & Ux),* **V**; (b) *(Ux)(Ix ⊃ Px), (Ux)(Ix ⊃ ~Jx) / ∴ ~(Ux)(Px ⊃ Jx),* **I**, *P/J/I = F/G/H, F6, 3=7=S, 2=3=S;* (c) *(Ux)(Sx ⊃ Wx), (Ux)(Rx ⊃ ~Wx) / ∴ (Ux)(Sx ⊃ ~Rx),* **V**; (d) *(Ux)(Px ⊃ Ix), (Ux)(Px ⊃ Mx) / ∴ (Ex)(Mx & Ix),* **I**, *P/I/M = F/G/H, F6, 5=6=S, 1=5=S;* (e) *~(Ex) (Ax & Sx), (Ex)(Sx & Px) / ∴ (Ex)(Px & ~Ax),* **V**; (f) *(Ux)(Ix ⊃ Sx), (Ex)(Bx & ~Sx / ∴ (Ex)(Bx & ~Ix),* **V**; (g) *(Ex)(Ax & Hx), (Ux)(Fx ⊃ Hx) / ∴ (Ex)(Ax & Fx),* **I**, *A/H/F = F/G/H, F6, 1=x, 6=7=S;* (h) *(Ux)(Tx ⊃ Ex), (Ux)(Ex ⊃ Cx) / ∴ (Ux)(Tx ⊃ Cx),* **V**; (i) *~(Ex)(Rx & Ax), (Ux)(Dx ⊃ Rx) / ∴ ~(Ex)(Dx & Ax),* **V**; (j) *(Ux)(Lx ⊃ ~Ex), (Ux)(Ix ⊃ Lx) / ∴ (Ex)(Ex & ~Ix),* **I**, *L/E/I = F/G/H, F6, 1=2=S, 3=7=S;* (k) *(Ux)(Rx ⊃ Fx), (Ux)(Fx ⊃ ~Bx) / ∴ (Ex)(Rx & ~Bx),* **I**, *R/B/F = F/G/H, F6, 5=1=S, 2=3 = S;* (l) *(Ux)(Bx ⊃ Fx), (Ux)(Hx ⊃ Fx) / ∴ (Ex)(Bx & Hx),* **I**, *B/F/H = F/G/H, F6, 5=6=S, 6=7=S;* (m) *(Ex) Sx, ~(Ex) Rx / ∴ (Ex)(Sx & ~Rx),* **V**; (n) *(Ux)(Sx ⊃ Bx), (Ux)(Bx ⊃ Px) / ∴ (Ux)(Px ⊃ Sx),* **I**, *P/S/B = F/G/H, F6, 1=4=S, 3=7=S;* (o) *(Ux)(Tx ⊃ Sx), ~(Ex)(Wx& Sx) / ∴ (Ex)(Wx & ~ Tx),* **I**, *T/S/W = F/G/H, F6, 5=6=S, 2=3=S;* (p) *~ (Ex)(Ax & Ix), (Ex)(Ax & ~ Wx) / ∴ ~ (Ex)(Ix & Wx)* **I**, *A/I/W = F/G/H, F6, 1=2=S, 1–5=x, 3=x;* (q) *~ (Ex)(Ax & Sx), (Ux)((Ax ∨ Sx) ⊃ Px), (Ux)((Px & Tx) ⊃ Ax) / ∴ (Ux)(Sx ⊃ (Px & ~Tx)),* **V**; (r) *~ (Ex)(Bx & Fx), (Ux)(Mx ⊃ Rx), (Ux)(Rx ⊃ Fx) / ∴ ~ (Ex)(Mx & Bx & Px),* **V**; (s) *~ (Ex)(Ex & Ix), (Ex)(Ox & Ix) & (Ex)(Tx & Ix) / ∴ (Ex)(Ox & ~ Ex) & (Ex)(Tx & ~ Ex),* **V**; (t) *(Ux)((Px & Tx & Ax) ⊃ ~Sx), (Ex)(Px & Ax & Hx & Tx) / ∴ (Ex)(Px & Hx & ~ Sx),* **V**.

Ex 13.4

1. (a) *(Ux)(Nx ⊃ Px), (Ux)(Px ⊃ ~ Rx) / ∴ (Ux)(Nx ⊃ ~ Rx), F6, N/R/F=F/G/H; 1 & 5, 2 & 3 / ∴ 1 & 2,* **V**; (b) *(Ux)(Lx ⊃ Gx), (Ux)(Gx ⊃ Px) / ∴ (Ex)(Px & Lx), F6, L/G/P = F/G/H, 5 & 6, 1 & 4 / ∴ ~ 2 ∨ ~6,* **I**; (c) *(Ux)(Dx ⊃ ~Ux), (Ux)(Ax ⊃ Dx) / ∴ (Ex)(Ux & ~ Dx), F6, D/U/A = F/G/H, 1 & 2, 6 & 7 / ∴ ~ 3 ∨ ~ 4,* **I**; (d) *(Ex)(Rx & ~ Fx), (Ux)(Rx ⊃ Lx) / ∴ (Ex)(Lx & ~Fx), F6, R/F/L = F/G/H, ~ 5 ∨ ~ 6, 1 & 5 / ∴ ~ 6 ∨ ~ 7,* **V**; (e) *(Ex)(Sx & Hx), (Ex)(Sx & ~ Bx) / ∴ (Ex)(Hx & ~ Bx), F6, S/H/B = F/G/H, ~1 ∨ ~2, ~ 1 ∨ ~ 5 / ∴ ~1 ∨ ~4,* **I**; (f) *(Ux)(Mx ⊃ (Ox & Gx)), (Ux)(Ox ⊃ Fx), (Ux)((Gx ∨ ~Fx) ⊃ Px), (Ux)((Px & Fx) ⊃ ~Gx), (Ex)(Mx & (Fx ≡ Ox)) / ∴ ~(Ux)(Mx ⊃ Fx), M/O/G/F/P = F/G/H/I/J, F8x2 (LH = J: 1–16; RH = J′: 17–32), 1 & 5 & 9 & 10 & 13 & 14 & 17 & 21 & 25 & 26 & 29 & 30, 1 & 2 & 3 & 4 & 17 & 18 & 19 & 20, 17 & 18 & 19 & 20 & 22 & 23 & 26 & 27 & 29 & 30 & 31 & 32, 6 & 7 & 10 & 11, ~5 ∨ ~ 6 ∨ ~13 ∨ ~14 ∨ ~21 ∨ ~22 ∨ ~29 ∨ ~30 / ∴ ~ 1 ∨ ~2 ∨ ~13 ∨ ~14 ∨ ~17 ∨ ~18 ∨ ~29 ∨ ~30,* **V**; (g) *R ⊃ (Ux)((Qx ∨ Dx) ⊃ Ex), ~(Ux)(Qx ∨ Dx) ⊃ Ax) / ∴ R ⊃ ~(Ux)(Ex ⊃ Ax), Q/D/E/A = F/G/H/I, F8, R ⊃ (1 & 5 & 9 & 13 & 4 & 8) ~ (1 & 2 & 3 & 4 & 13 & 14) / ∴ R ⊃ ~(2 & 3 & 14 & 15),* **V**; (h) *(Ux)((Ax & Wx & ~Hx) ⊃ Dx), (Ex)(Wx & Hx), (Ux)(Ax ⊃ Wx) / ∴ (Ex)(Ax & Dx), A/W/D/H = F/G/H/I, F8, 1, ~5 ∨ ~ 6 ∨ ~7 ∨ ~8, 9 & 10 & 13 & 14 / ∴ ~ 2 ∨ ~6 ∨ ~10 ∨ ~14,* **I**; (i) *(I & (Ex)(Tx & ~ Rx)) ⊃ (Ex)(Sx & ~Bx), (Ux)(Fx ⊃ Bx), (I & (Ex)(Sx & ~Fx) / ∴ ~(Ux)(Tx ⊃ Rx), T/R/S/B/F = F/G/H/I/J, F8x2, (I & ~(9 & 10 & 13 & 14 & 25 & 26 & 29 & 30)) ⊃ ~(2 & 3 & 14 & 15 & 18 & 19 & 30 & 31), 1 & 2 & 3 & 4 & 13 & 14 & 15 & 16, I & ~ (18 & 19 & 22 & 23 & 26 & 27 & 30 & 31) / ∴ ~ (9 & 10 & 13 & 14 & 25 & 26 & 29 & 30),* **I**; (j) *(Ux)(Ex ⊃ Ix) ∨ ~(Ex)(Px & Rx) / ∴ (Ex)(Px & Ex) ⊃ (Ex)(Ex & (Ix ∨ ~Rx)), E/I/P/R = F/G/H/I, F8, (9 & 10 & 13 & 14) ∨ (6 & 7 & 10 & 11) / ∴ ~ (2 & 6 & 10 & 14) ⊃ ~(1 & 2 & 5 & 6 & 13 & 14),* **V**.

Puzzle 13 In clockwise order: Henry, Anne, Fred, Beth, Cath, George.

Ex 14.1

1. (a) 2 (b) 2 (c) 2 (d) 3 (e) 4
2. (a) Active. (b) Passive. The ball struck Mike. (c) Passive. The umpire declared Dennis out. (d) Active. (e) Passive. Given any person, someone will respect that person.
3. (a) **I** active. (b) **O** active (c) **I** passive (d) **E** over **A** active (e) **A** over **I** passive (f) **I** over **A** passive (g) **I** over **A** passive (h) **A** by **I** passive over **A** active (i) **E** by **I** active over **A** active (j) **A** by **E** active active.

Ex 14.2

1. Rules used: (a) B2, B2, R⊃, RU (b) B2, RE, B1, R⊃ (c) See solution below. (d) B2, RU, RE, RE, R ∼, B2, B2, R&, R ∼, RE, RU, R⊃ (e) B1, B2, RE, R⊃, B2, R ∼, R∨, RU.

Solution to (c):

1. F^2xy B2
2. $(\exists y)F^2xy$ 1, R∃
3. $(\exists x)(\exists y)F^2xy$ 2, R∃
4. G^1a B2
5. H^1b B2
6. $(G^1a \supset H^1b)$ 4, 5, $R\supset$
7. $(\forall x)(G^1a \supset H^1b)$ 6, $R\forall$
8. $((\exists x)(\exists y)F^2xy \equiv (\forall x)(G^1a \supset H^1b))$ 3, 7, R≡

Ex 14.3

1. (a) (i) $(Pa \supset aRa) \& (Pb \supset bRa)$: *O* (ii) $Pa \supset aRa$; *1* (b) (i) $(Pa \supset \sim aRa) \& (Pb \supset \sim aRb)$:*1* (ii) $Pa \supset \sim aRa$: *1* (c) (i) $[(Pa \& Ga) \& aRa] \vee [(Pb \& Gb) \& bRa]$: *O* (ii) $(Pa \& Ga) \& aRa$:*O* (d) (i) $[(La \& Ga) \supset aRa] \& [(Lb \& Gb) \supset bRa]$: *1* (ii) $(La \& Ga) \supset aRa$: *O* (e) (i) $(aRa \vee aRb) \& (bRa \vee bRb)$: *1* (ii) aRa : *O* (f) (i) $(\sim aRa \& \sim bRa) \vee (\sim aRb \& \sim bRb)$: *O* (ii) $\sim aRa$: *1* (g) (i) $[Pa \supset ((Pa \& Ga \& \sim aRa) \vee (Pb \& Gb \& \sim aRb))] \& [Pb \supset ((Pa \& Ga \& \sim bRa) \vee (Pb \& Gb \& \sim bRa) \vee (Pb \& Gb \& \sim bRb))]$: *O* (ii) $Pa \supset (Pa \& Ga \& \sim aRa)$: *1* (h) (i) $[(La \& (Ga \supset aRa) \& (Gb \supset aRb)) \supset aRa] \& [(Lb \& (Ga \supset bRa) \& (Gb \supset bRb)) \supset bRa]$: *O* (ii.) $(La \& (Ga \supset aRa)) \supset aRa$: *1* (i) (i) $(aRa \vee bRa) \supset p$: *O* (ii) $aRa \supset p$: *1* (j) (i) $[p \supset (aRa \vee bRa)] \& [p \supset (aRb \vee bRb)]$: *1* (ii) $p \supset aRa$: *O*

Ex 14.4

1. (a) C^+ applies to (c) and (d). QT- Necessities: b, d, e, f, g, i
2. (c) is not a QT-Necessity.
3. (g)

✓ 1. $\sim[(\exists x)(\forall y)xAy \supset [(\forall x)(\forall y)(yAx \supset yBx) \supset (\forall y)(\exists x)xBy]]$ NF
✓ 2. $(\exists x)(\forall y)xAy$ } 1, PC
✓ 3. $\sim[(\forall x)(\forall y)(yAx \supset yBx) \supset (\forall y)(\exists x)xBy]$
ba : *xy* \ 4. $(\forall x)(\forall y)(yAx \supset yBx)$ } 3, PC
✓ 5. $\sim(\forall y)(\exists x)xBy$
✓ 6. $(\exists y)(\forall x)\sim xBy$ 5, QN × 2
b \ 7. $(\forall y)aAy$ 2, EI
a \ 8. $(\forall x)\sim xBb$ 6, EI
9. aAb 7, UI
10. $\sim aBb$ 8, UI
✓ 11. $aAb \supset aBb$ 4, UI × 2
12. $\sim aAb$ aBb 11, PC
× ×

The tree closes. Therefore the formula is QT-necessary.

4. Sample counterexamples. (a) $aRa = bRb = 1, aRb = bRa = 0$ (b) $bRa = 0, rest = 1$ (c) $aRa = 1, rest = 0$ (d) $aRa = 1, rest = 0$ (e) $Fa = aRb = 1, rest = 0$
5. QTSCC: a, c, d. (a) Valid (c) Invalid (d) Invalid.
6. (b) Valid (e) Invalid

7. (f)

$abc : xyz \setminus$	1	$(\forall x)(\forall y)(\forall z)[xGy \,\&\, xGz \,.\!\supset\!.\, Bxyz]$			P
✓	2.	$(\exists x)[Nx \,\&\, (\exists y)(\exists z)(xGy \,\&\, xGz)]$			P
$abc : xyz \setminus$	3.	$(\forall x)(\forall y)(\forall z)(Bxyz \supset \sim Nx)$			NC
✓	4.	$Na \,\&\, (\exists y)(\exists z)(aGy \,\&\, aGz)$			2, EI
	5.	Na			4, PC
✓	6.	$(\exists y)(\exists z)(aGy \,\&\, aGz)$			
✓	7.	$aGb \,\&\, aGc$			6, EI×2
✓	8.	$aGb \,\&\, aGc \,.\!\supset\!.\, Babc$			1, UI×3
	9.	$\sim(aGb \,\&\, aGc)$ ×	$Babc$		8, PC
✓	10.		$Babc \supset \sim Na$		3, UI×3
	11.		$\sim Babc$ ×	$\sim Na$ ×	10, PC

Closure. ∴ Valid

8. Counterexamples: a, b, d

9. (a) $aRa = 1, rest = 0$ (b) $aRa = 1, rest = 0$ (c) $aRb = 1, rest = 0$
(d) $aRa = bRb = aSa = bSb = 1, rest = 0$ (e) $p = Fb = bRa = 0, rest = 1$

Ex. 14.5

Note: for ease of typesetting we use U for $\forall$ and E for $\exists$.

1. (a) Se (b) $Sm \,\&\, Ss \,\&\, Se$ (c) sHm (d) $\sim eHs \,\&\, \sim mHs$ (e) $Ss \,\&\, Sm \,\&\, sHm$ (f) $(Ex)\,(Sx \,\&\, xHm \,\&\, xHe)$ (g) $\sim (Ex)(Sx \,\&\, xHs)$ (h) $sHm \vee eHm$ (i) $(Ux)(Sx \,\&\, xHs \,.\!\supset\!.\, xHe)$ (j) $(Ex)(Sx \,\&\, xHm) \,\&\, (Ex)(Sx \,\&\, \sim xHs)$ (k) $(Ux)[(Sx \,\&\, xHe) \supset (Sx \,\&\, xHm)]$
2. (a) $(Ux)(Px \supset xRb)$ (b) $(Ex)(Px \,\&\, xRb)$ (c) $(Ux)(Px \supset \sim xRb)$
(d) $(Ex)(Px \,\&\, \sim xRb)$ (e) $(Ux)([(Px \,\&\, xRb) \supset Gx]$
(f) $(Ux)[(Px \,\&\, \sim xRb) \supset Gx]$ (g) $(Ex)(Px \,\&\, xRb \,\&\, \sim Gx)$ (h) $(Ex)(Px \,\&\, Gx \,\&\, \sim xRb)$
(i) $(Ux)[(Px \,\&\, Gx) \supset xRb]$ (j) $(Ux)[(Px \,\&\, xRc) \supset xRb)$ (k) $(Ux)[(Px \,\&\, xRc) \supset (Gx \vee xRb)]$
3. (a) $bHc \,\&\, \sim cHd$ (b) $(Ux)(Px \supset xHc)$ (c) $(Ux)(Px \supset cHx)$
(d) $(Ux)[Px \supset (Ey)(Py \,\&\, xHy)]$ (e) $(Ux)[Px \supset (Ey)(Py \,\&\, yHx)]$
(f) $(Ux)[(Px \,\&\, xHd) \supset Kx]$ (g) $(Ux)[(Px \,\&\, dHx) \supset Dx]$
(h) $(Ux)[(Px \,\&\, (Ey)(Py \,\&\, xHy)) \supset Kx]$ (i) $(Ux)[(Px \,\&\, (Ey)(Py \,\&\, yHx)) \supset Dx]$
(j) $(Ux)[(Px \,\&\, Kx) \supset (Ey)(Py \,\&\, xHy)]$ (k) $(Ux)[(Px \,\&\, Dx) \supset (Ey)(Py \,\&\, yHx)]$
4. (a) $\sim mCb$ (b) $(Ex)\, bCx$ (c) $\sim (Ux)\, mCx$ (d) $(Ux)(Px \supset \sim xCj)$ (e) $(Ux)(Px \supset dLx)$
(f) $\sim (Ex)(Px \,\&\, xCd)$ (g) $(Ux)[Px \supset \sim (Ey)\, xCy]$ (h) $(Ex)(Ey)(\sim Px \,\&\, Py \,\&\, xLy)$
(i) $mCj \,\&\, \sim (Ux)(Px \supset xCj)$
5. (a) $\sim eLm \,\&\, eIm$ (b) $\sim (Ex)(Px \,\&\, eLx) \supset \sim eLe$ (c)$(Ux)[Px \,\&\, (Uy)(Py \supset xLy) \,.\!\supset\!.\, Gx]$
(d) $(Ex)(Px \,\&\, Rxme) \supset eIe$ or $(Ux)[(Px \,\&\, Rxme) \supset eLe]$ (e) $aLm \,\&\, mLa \,\&\, Rame$
6. (a) aPb (b) $(Ex)(Hx \,\&\, \sim xPb)$ (c) $Ma \,\&\, aPb$ (d) $(Ex)(Mx \,\&\, Hx \,\&\, aPx)$
(e) $Hb \,\&\, \sim Mb$ (f) $Hb \,\&\, \sim Mb \,\&\, aPb$ (g) $(Ex)\, xPb$ (h) $(Ex)(cPx \,\&\, xPb)$
(i) $(Ex)(Ey)(xPy \,\&\, yPb)$ (j) $(Ex)(Ey)(\sim Mx \,\&\, xPy \,\&\, yPb)$ (k) $(Ux)(Hx \supset (Ey)\, yPx)$
(l) $\sim (Ux)[(Ey)\, xPy \supset Mx]$ (m) $(Ux)[Mx \,\&\, (Ey)\, xPy \,.\!\supset\!.\, Mx]$ (n) $\sim (Ex)(Hx \,\&\, xPx)$
7. (a)Pb (b) $(Ux)(Px \supset Hx)$ (c) $Pc \supset Pk$ (d) $(Ex)[Px \,\&\, (Uy)(Py \supset yLx)]$
(e) $(Ux)[Px \supset (Ey)(Py \,\&\, xLy)]$ (f) same as (e) (g) $(Ex)(Mx \,\&\, xLx)$
(h) $(Ex)\, Px \supset Pb$ (i) $(Ux)[(Lx \vee Px) \supset (Ix \,\&\, Sx)]$ (j) $\sim(Ex)\, Mx$ (k) $(Ex)(Px \,\&\, Hx \,\&\, \sim Lx)$
(l) $(Ux)(Wx \supset Lx)$ (m) For this and the next translation, $Lx = x$ is a case of laziness. $(Ux)(Sx \supset \sim Lx)$ (n) $(Ex)(Lx \,\&\, Dx)$ (o) $(Ux)(Sx \supset Mx)$ (p) $(Ux)(Tx \supset Gx)$ (q) $(Ux)(Cx \,\&\, Hx \,\&\, Sx \,\&\, Bx \,.\!\supset\!.\, Dx)$
8. (a) Not everything is a sport or mental activity. (b) Each person is good at some sport. (c) Not everyone is good at all mental activities. (d) Everyone is good at something which is neither a sport nor a mental activity.

9. (a) Not all adult people are male. (b) Every man is loved by some woman or other. (c) Some adult person loves some adult person but is not loved by the latter. (d) There is a male who loves himself but loves no females. (e) If 2 adult people are either male or female but not both, then no adult person is both male and female.
10. (a) All positive numbers are real numbers. (b) There is a real number which is neither positive nor negative. (c) Given any two real numbers, either the first is greater than the second or the second is greater than the first or they are equal. (d) Given any positive number, there is no negative number greater than it. (e) Any real number greater than a positive number is itself positive.

Ex 14.6

Code: V = Valid; I = Invalid.

1. (a) V (b) V (c) V (d) V (e) V (f) V (g) V. As it stands, the argument has a countermodel, but this is an impossible model (and hence not a counterexample) since all vacant allotments are real estate (this further condition may be added as a tacit premise). (h) V (i) I (j) V (k) I (l) I (m) V (n) I (o) The symbolized argument is Valid, but it is debatable whether the original English argument is valid: it depends on whether the denial of the conclusion's antecedent implies that the conclusion is true. (p) V (q) V (r) V (assuming that Mr. Brown is a person) (s) V
2. (a) I (b) V (c) I (d) I (e) V (f) No. the argument is invalid. (g) V (h) V (i) V (j) V (k) I (l) I (m) I (n) V

Sample details:

(d) Whether or not you agree that "beliefs" as such do exist, this argument may be evaluated by taking beliefs as the "individuals" of a non-null universe of discourse.

$$(\forall x)(Rx \vee Ix \vee Ex)$$
$$(\forall x)(Ix \supset {\sim}Rx)$$
$$Ea$$
$$(\forall x)(Ix \supset {\sim}Jx)$$
$$\therefore\ {\sim}Ja$$

INVALID

	R	I	E	J
a	0	0	1	1

(j)

m .... Mars

Rx .... x is red

Px .... x is a planet

Ax .... x is an astronaut

xLy .... x lands on y

$$Rm \,\&\, Pm$$
$$\therefore\ (\forall x)[(Ax \,\&\, xLm) \supset (\exists y)(Py \,\&\, xLy)]$$

✓	1.	$Rm \,\&\, Pm$	P
✓	2.	${\sim}(\forall x)[(Ax \,\&\, xLm) \supset (\exists y)(Py \,\&\, xLy)]$	NC
✓	3.	$(\exists x){\sim}[(Ax \,\&\, xLm) \supset (\exists y)(Py \,\&\, xLy)]$	2, QN
	4.	Rm	1, PC
	5.	Pm	
✓	6.	${\sim}[(Aa \,\&\, aLm) \supset (\exists y)(Py \,\&\, aLy)]$	3, EI
✓	7.	$Aa \,\&\, aLm$	6, PC
✓	8.	${\sim}(\exists y)(Py \,\&\, aLy)$	
	9.	Aa	7, PC
	10.	aLm	
m \	11.	$(\forall y){\sim}(Py \,\&\, aLy)$	8, QN
✓	12.	${\sim}(Pm \,\&\, aLm)$	11, UI
	13.	${\sim}Pm$ × ${\sim}aLm$ ×	12, PC

∴ Valid

Puzzle 14 The destroyer is in status III in sector A51. The tanker is in status I in sector A52. The cruiser is in status II in sector A54. The scout is in status IV in sector A53.

Ex 15.1

1. (a) 1.P, 2. 1DN, 3.2 QN, 4.3 DeM, 5.4 MI; (b) 1.P, 2.1 DN, 3.2 QN, 4.3 MI, 5.4 DeM, 6.5 DN, 7.6 QN, etc.; (c) 1. P, 2.1 Contra.; (d) 1.P, 2.1 Contra, 3.2 DN; (e) 1.P, 2.1 DN, 3.2 QN.

Ex 15.2

1. Line numbers will not be used: (a) P, UI, EG; (b) P, P, UI, AA, Conj, EG; (c) P, P, UI, Simp, AA, Simp, Conj, EG; (d) P, P, QN, UI, Simp, Simp, DeM, DN, DD, Conj, EG; (e) P, P, Simp, Simp, UI, DC, Conj, EG; (f) P, UI wrt a, UI wrt b, EG, EG; (g) P, UI wrt a, UI wrt a, EG; (h) P, UI, EG; (i) P, P, P, EG (2), ..., DD, QN, UI, ...; (j) P, UI, Ap, AA, EG, CP.

Ex 15.3

1. (a) Correct; (b) Incorrect, every a → y; (c) Incorrect, a in P; (d) Incorrect, a in A; (e) Incorrect, a in P.
2. Hints: (a) QN, UI, ..., ChA, UG; (b) UI, MI, Add, DeM, ...; (d) Ap; (e) Ap; (f) A(Ex) Fx; (g) UI*x, z* wrt a; (h) UIx2 wrt a; (i) Add, MI

Ex 15.4

1. (a) Correct; (b) Incorrect, a in 2; (c) Incorrect, b in 2; (d) Correct; (e) Incorrect, a in 1.
2. Hint: (a) A(Ux)Fx before EI; (b) etc over to you.

Puzzle 15 1st: Corporal, arms, RAAF; 2nd: Army S, Bldg St, Army; 3rd: RAAFS, fuel, Navy; 4th: priv, food, Civ – so third is bomb truck.

BIBLIOGRAPHY

This bibliography lists the works referred to in this text. Some of the entries are annotated. The symbols used are:

B = Background
H = Historical
A1 or A2 = Appendix 1 or 2
(where *n* is a numeral)
P = Part one (PC)
Q = Part two (QT)
PS = Postscript
n = of particular relevance to Chapter *n*

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